On using convolutions with exponential distributions for solving a Kolmogorov backward equation

Vasily Rodochenko

Southern Federal University

Voronezh, december 2018

Introduction

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A diffusion equation

A 3-dimensional partial differential equation for function $u = u(x,t)$, where $x = (x_1, x_2) \in \mathbb{R}^2$, $t \in \mathbb{R}$:

General form (Øksendal)

$$\left(\frac{\partial}{\partial_t} + L\right) u = 0, \qquad L = \sum_i \mu_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j},$$

$i = 1, 2$, $j = 1, 2$; functions $\mu(x) = (\mu_1(x), \mu_2(x))$ and

$$\sigma(x) = \begin{pmatrix} \sigma_{11}(x) & \sigma_{12}(x) \\ \sigma_{21}(x) & \sigma_{22}(x) \end{pmatrix}$$

satisfies conditions of Theorem 5.2.1. $\mu : \mathbf{R}^2 \to \mathbf{R}^2$ — drift coefficient
$\sigma : \mathbf{R}^2 \to \mathbf{R}^2 \times \mathbf{R}^2$ (or $\frac{1}{2}(\sigma\sigma^T)$) — diffusion coefficient

- With a suitable initial and boundary conditions, the Kolmogorov backward equation can be solved (Øksendal; Eström (2010)).
- There are no general analytical solutions.
- Numerical methods are based on the exact form of $\mu_i(x)$ and $\sigma(x)$.

Let $T > 0$ be a time moment, $H > 0$ – an absorbing barrier, $g(x) : \mathbf{R} \to \mathbf{R}^{\geqslant 0}$ – some suitable function, which decays rapidly on infinity.

Terminal and boundary conditions

$$\begin{cases} (\frac{\partial}{\partial t} + L)u = 0, & x_1 > H, t < T, \\ \\ u(x_1, x_2, T) = g(x_1), & x_1 > H, \\ u(x_1, x_2, t) = 0, & x_1 \leqslant H, t \leqslant T. \end{cases}$$

A diffusion equation

- The solution to a diffusion equation in a given domain can be interpreted as an expectation (Dynkin, Feinman, Katz).
- The idea was later generalized for the case of Lévy processes.

Ito diffusions - general

Two correlated diffusions

$$\begin{cases} dX_1(t) = \mu_1 dt + \sigma_{11} dB_1(t) + \sigma_{12} dB_2(t), \\ dX_2(t) = \mu_2 dt + \sigma_{21} dB_1(t) + \sigma_{22} dB_2(t). \end{cases}$$

- $B_1(t), B_2(t)$ are Brownian motions (Wiener processes)

An example from mathematical finance

Define const. $\kappa_V > 0, \theta_V > 0$, $\sigma_V > 0$; $\rho \in (-1, 1)$, $\hat{\rho} = \sqrt{1 - \rho^2}$.
Denote $X_1(t)$ as S_t and $X_2(t)$ as V_t. Assume $\mu_1 = 0$,
$\mu_2 = \kappa_V(\theta_V - V_t)$ and

$$\sigma = \begin{pmatrix} \rho\sqrt{V_t}S_t & \hat{\rho}\sqrt{V_t}S_t \\ \sigma_V\sqrt{V_t} & 0 \end{pmatrix},$$

Processes:

$$\begin{cases} dS_t = \sqrt{V_t}S_t(\rho dB_1(t) + \hat{\rho}dB_2(t)), \\ dV_t = \kappa_V(\theta_V - V_t)dt + \sigma_V\sqrt{V_t}dB_1(t). \end{cases} \quad (1)$$

An infinitesimal operator L:

$$L = \frac{1}{2}S^2 v\frac{\partial^2}{\partial S^2} + \rho\sigma_V vS\frac{\partial^2}{\partial S\partial v} + \frac{1}{2}\sigma_V^2 v\frac{\partial^2}{\partial v^2} + \kappa_V(\theta_V - v)\frac{\partial}{\partial v}. \quad (2)$$

The problem in u(S, v, t) terms

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{1}{2}S^2 v \frac{\partial^2}{\partial S^2} + \rho\sigma_V vS \frac{\partial^2}{\partial S \partial v} + \right. \\ \left. + \frac{1}{2}\sigma_V^2 v \frac{\partial^2}{\partial v^2} + \kappa_V(\theta_V - v)\frac{\partial}{\partial v}\right) u = 0, & S > H, v > 0, t < T, \\ u(S, v, T) = g(S), & S > H, v > 0, \\ u(S, v, t) = 0, & S \leqslant H, v > 0, t \leqslant T. \end{cases}$$

The solution (exist. & uniq. — Cont, Tankov (2004)):

$$u(S, v, 0) = E[\mathbf{1}_{(T,+\infty)}(T_H) \cdot g(S_T) | S_0 = S, V_0 = v], \qquad t = 0. \quad (3)$$

The substitution (Zanette, Briani et al. (2017))

The structure:

$$Y_t = \ln\left(\frac{S_t}{H}\right) - \frac{\rho}{\sigma_V} V_t, \qquad S_t = H\exp\left(Y_t + \frac{\rho}{\sigma_V} V_t\right).$$

The system in its terms:

$$\begin{cases} dY_t = \mu_Y(V_t)dt + \hat{\rho}\sqrt{V_t}dB_2(t), \\ dV_t = \mu_V(V_t)dt + \sigma_V\sqrt{V_t}dB_1(t), \end{cases}$$

where

$$\mu_Y(v) = -\frac{1}{2}v - \frac{\rho}{\sigma_V}\kappa_V(\theta_V - v)$$

and

$$\mu_V(v) = \kappa_V(\theta_V - v).$$

With $g(y) = g(He^{y+\frac{\rho}{\sigma}v})$:

The problem in terms of $f(y, v, t) := u\Big(H\exp(y + \frac{\rho}{\sigma_V}v), v, t\Big)$

$$\begin{cases} \left(\frac{\partial}{\partial_t} + \frac{1}{2}\hat{\rho}^2 v \frac{\partial^2}{\partial y^2} \right. & \left. + \frac{1}{2}\sigma_V^2 v \frac{\partial}{\partial v^2} + \mu_Y(v)\frac{\partial}{\partial y} + \mu_V(v)\frac{\partial}{\partial v}\right) f = 0, \\ & y + \frac{\rho}{\sigma_V}v > 0, v > 0, t < T, \\ f(y, v, T) = g(y), & y + \frac{\rho}{\sigma_V}v > 0, v > 0, \\ f(y, v, t) = 0, & y + \frac{\rho}{\sigma_V}v \leqslant 0, v > 0, t \leqslant T. \end{cases}$$

T_H is as the earliest time the process $Y_t + \frac{\rho}{\sigma}V_t$ enters $(-\infty, 0]$:

$$T_H = \inf_{t \geqslant 0}\{t : Y_t + \frac{\rho}{\sigma_V}V_t \leqslant 0\}.$$

The expectation

$$f(y, v, 0) = E^{y,v}[\mathbf{1}_{(T,+\infty)}(T_H)g(Y_T)].$$

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Carr randomization

Let $N \in \mathbb{N}$, $\Delta t = T/N$, $q > 0$, $T_q \sim Exp(\Delta t^{-1})$.
We calculate a sequence of

$$f_n(y,v) \approx f(y,v,\sum_{j=1}^{n} T_q^j), \qquad n = 0,1,\ldots,N;$$

where $f_0(y,v) \approx f(y,v,0)$; $\quad f_N(y,v) = g(y)$,
and $T_q^j \sim Exp(\Delta t^{-1})$ is a sequence of ind. random variables.

Expectations

$$f_n(y) = E^{y,v}\Big[\mathbf{1}_{T_q^{n+1},+\infty)}\big(\inf_{t\geqslant 0}\{t : Y_{T_q^{n+1}} + \frac{\rho}{\sigma_V} V_{T_q^{n+1}} \leqslant 0\}\big) \cdot \cdot f_{n+1}(Y_{T_q^{n+1}}, V_{T_q^{n+1}})\Big]$$

The Markov chain

The recombining binomial tree:

$$V_{n,k} = \left(\sqrt{V_0} + \frac{\sigma_V}{2}(2k-n)\sqrt{\Delta t}\right)^2 \cdot \mathbf{1}_{(0,+\infty)}\left(\sqrt{V_0} + \frac{\sigma_V}{2}(2k-n)\sqrt{\Delta t}\right),$$

where $n = 0, 1, \ldots, N,\ k = 0, 1, \ldots, n.$

Transitions

From (n, k) to either $(n+1, k_u)$ or $(n+1, k_d)$.
With probabilities p_u and p_d.
The values k_u, k_d and p_u, p_d are based V_t, like in (Briani, Zanette at al (2013)).

Expectations

Recurrent calculation w.r.t the Markov chain:

$$f_{n,k}(y) = \mathbf{1}_{(0,+\infty)}\big(y + \frac{\rho}{\sigma_V} V_{n,k}\big) \cdot \big(p_u f^u_{n,k}(y) + p_d f^d_{n,k}(y)\big),$$

where

$$f^u_{n,k}(y) = E^y\big[\mathbf{1}_{(0,+\infty)}(\underline{Y}^{n+1,k_u}_{T_q} + \frac{\rho}{\sigma_V} V_{n+1,k_u}) \cdot f_{n+1}(Y^{n+1,k_u}_{T_q}, V_{n+1,k_u})\big],$$

and $Y^{n,k}_{T_q}$ is a value of Y_{T_q} from the system, when $V_{T_q} = V_{n,k}$;

the value of $f^d_{n,k}(y)$ is calculated analogously.

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Expected Present Value (S. Levendorskiĭ) operator

$$\mathcal{E}_q F(x) = qE^x\left[\int_0^{+\infty} e^{-qt} F(X_t)dt\right] = E\left[F(x + X_{T_q})\right],$$

where X_t is an Ito diffusion.

Sup and inf processes with the EPVs

$$\overline{X_t} = \sup_{0\leqslant s\leqslant t} X_t, \quad \underline{X_t} = \inf_{0\leqslant s\leqslant t} X_t,$$

$$\mathcal{E}_q^+ F(x) = qE^x[\int_0^{+\infty} e^{-qt} F(\overline{X}_t)dt] = E[F(x + \overline{X}_{T_q}],$$

$$\mathcal{E}_q^- F(x) = qE^x[\int_0^{+\infty} e^{-qt} F(\underline{X}_t)dt] = E[F(x + \underline{X}_{T_q})].$$

EPV in the problem considered

Using the property $\overline{X}_t \overset{d}{\sim} X_t - \underline{X}_t$, one can quite easily obtain:

The solution in terms of the EPV operators

$$f^u_{n,k}(y) = q\mathcal{E}^-_q\left(\mathbf{1}_{(0,+\infty)}(y + \frac{\rho}{\sigma_V}V_{n+1,k_u}) \cdot \mathcal{E}^+_q f_{n+1,k_u}(y)\right),$$

and use an analogous formula for $f^d_{n,k}(y)$.

EPV operators as convolutions

$$\mathcal{E}_q F(x) = \int_{-\infty}^{+\infty} F(x+u)P_q(du), \quad \mathcal{E}^\pm_q F(x) = \int_{-\infty}^{+\infty} F(x+u)P^\pm_q(du),$$

where $P^+_q(-\infty, 0) = 0, \quad P^-_q(0, +\infty) = 0.$

Wiener-Hopf factorization for a symbol of a PDO

If X_t is a Lévy process, and its char. exp. is a function $\psi(\xi)$, defined by the Levy-Khintchine formula, then the char. functions for the distributions $P_q^{\pm}(du)$ are $\phi^{\pm}(\xi)$, such that:

$$\phi^+(\xi) \cdot \phi^-(\xi) = q(q + \psi(\xi))^{-1}. \quad (4)$$

There are also exist constants $\omega_- < 0 < \omega_+$ such that $\phi^+(\xi)$ admits an analytical cont. into half-plane $\mathrm{Im}\,\xi > \omega_-$ (and $\phi^-(\xi)$ — into $\mathrm{Im}\,\xi < \omega_+$.

PDO representation and symbols

$q(q + \psi(\xi))^{-1}$ is a symbol of $\mathcal{E}_q$
$\phi^{\pm}(\xi)$ are symbols of $\mathcal{E}_q^{\pm}$.

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The characteristic function of $Y_t^{n,k}$, $t < T_q$, is

$$\psi(\xi) = \frac{\sigma_{n,k}^2}{2}\xi^2 - i\gamma_{n,k}\xi,$$

where $\sigma_{n,k} = \hat{\rho}\sqrt{V_{n,k}}$, $\gamma_{n,k} = \mu_Y(V_{n,k})$. For this case:

$$\phi^+(\xi) = \frac{\beta_q^+}{\beta_q^+ - i\xi}, \quad \phi^-(\xi) = \frac{-\beta_q^-}{-\beta_q^- + i\xi},$$

where

$$\beta_q^+ = \frac{-\gamma_{n,k} + \sqrt{\gamma_{n,k}^2 + 2\sigma_{n,k}^2 q}}{\sigma_{n,k}^2}, \qquad \beta_q^- = \frac{-\gamma_{n,k} - \sqrt{\gamma_{n,k}^2 + 2\sigma_{n,k}^2 q}}{\sigma_{n,k}^2}.$$

The respective distributions:

$$P_q^-(du) = -\beta_q^- e^{-\beta_q^- u}\mathbf{1}_{(-\infty,0)}(u)du, P_q^+(du) = \beta_q^+ e^{-\beta_q^+ u}\mathbf{1}_{(0,+\infty)}(u)du$$

.

PDO representation ($O\ n\log(n)$ operations via FFT):

$$\varepsilon_q^{\pm}u(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}\phi_q^{\pm}(\xi)\hat{u}(\xi)d\xi.$$

Let us define a reasonably dense grid $y_k = y^* + yh$, $k = 0, 1, \dots M$, where $M \in \mathbb{N}$ is large, denote $F_n(y) := f_{n,k}^u(y)$, $F_{n+1}(y) := f_{n+1,k_u}^u(y)$ and consider the equation:

Convolutions calculation

$$\mathcal{E}^+ F_{n+1}(y_k) = \int_0^{+\infty} \beta_q^+ e^{-\beta_q^+ u} F_{n+1}(y_k + u)du. \quad (5)$$

$\mathcal{E}^+$, step 1

As $g(y)$ decays on infinity, $\lim_{y \to +\infty} F_n(y) = \lim_{y \to +\infty} g(y) = 0$, we restrict the integration area with $y^* > 0$. If $y_k + u < y^*$, then $u < y^* - y_k$. We get:

$$\mathcal{E}^+ F_{n+1}(y_k) \approx \int_0^{y^* - y_k} \beta_q^+ e^{-\beta_q^+ u} F_{n+1}(y_k + u) du.$$

Make a substitution: $w = u + y_k$:

$$\mathcal{E}^+ F_{n+1}(y_k) \approx e^{\beta_q^+ y_k} \int_{y_k}^{y^*} \beta_q^+ e^{-\beta_q^+ w} F_{n+1}(w) dw.$$

$\mathcal{E}^+$, step 2

From an indicator: for $y_k - h < -\frac{\rho}{\sigma_V} V_{n+1,k_u}$, $F_{n+1}(y_k - h) = 0$.

$$\mathcal{E}^+ F_{n+1}(y_{k-1}) \approx e^{-\beta_q^+ h} \mathcal{E}^+ F_{n+1}(y_k) + e^{-\beta_q^+ h} \int_{y_k - h}^{y_k} \beta_q^+ e^{-\beta_q^+ (w - y_k)} F_{n+1}(w) dw.$$

Trapezoid approximation for the integral part:

$$\frac{h}{2} \cdot \beta_q^+ \left(F_{n+1}(y_{k-1}) + e^{-\beta_q^+ h} F_{n+1}(y_k) \right)$$

Simpson approximation example ($\mathcal{E}^+ F_{n+1}(y_{k-2})$ used):

$$\frac{h}{3} \cdot \beta_q^+ \left(F_{n+1}(y_{k-2}) + 4 e^{-\beta_q^+ h} F_{n+1}(y_{k-1}) + e^{-2\beta_q^+ h} F_{n+1}(y_k) \right)$$

$\mathcal{E}^-$, step 1

Let us put $z = \frac{\rho}{\sigma_V} V_{n+1,k_u}$ and consider:

$$q\mathcal{E}^-\left(\mathbf{1}_{(0,+\infty)}(y_k + z) \cdot \mathcal{E}^+ F_{n+1}(y_k)\right) =$$

$$= q \int_{-(z+y_k)}^{0} -\beta_q^- e^{-\beta_q^- u} \cdot \mathcal{E}^+ F_{n+1}(u + y_k) du$$

The integral form for $q\mathcal{E}^-$ in y_k:

$$\stackrel{w:=u+y_k}{=} q e^{\beta_q^- y_k} \int_{-z}^{y_k} -\beta_q^- e^{-\beta_q^- w} \mathcal{E}^+ F_{n+1}(w) dw.$$

$\mathcal{E}^-$, step 2

Then, for $y_{k+1} = y_k + h$, using the same reasoning we get:

$$q\mathcal{E}^-\left(\mathbf{1}_{(0,+\infty)}(y_{k+1}+z)\cdot\mathcal{E}^+F_{n+1}(y_{k+1})\right) =$$

$$= qe^{\beta_q^-(y_k+h)}\left(e^{\beta_q^- h}\mathcal{E}^-\mathbf{1}_{(-\infty,0)}(y_k+z) + \right.$$

$$\left. + \int_{y_k}^{y_k+h} -\beta_q^- e^{-\beta_q^- w}\mathcal{E}^+F_{n+1}(w)dw\right)$$

Trapezoid approximation

$$\frac{h}{2}\cdot(-\beta_q^-)\left(\mathcal{E}^+F_{n+1}(y_{k+1}) + e^{\beta_q^- h}\mathcal{E}^+F_{n+1}(y_k)\right)$$

Simpson approximation ($\mathcal{E}^-F_{n+1}(y_{k+2})$ used)

$$\frac{h}{3}\cdot(-\beta_q^-)\left(e^{2\beta_q^- h}\mathcal{E}^+F_{n+1}(y_{k+2}) + e^{\beta_q^- h}\mathcal{E}^+F_{n+1}(y_{k+1}) + \mathcal{E}^+F_{n+1}(y_k)\right)$$

Summary

- Using the procedure above to solve the arising 1-dimensional problems, we obtain an approximate solution for the Kolmogorov backward equation.
- The scheme presented is an iterative method for calculating integrals, where already calculated values of $\mathcal{E}^{\pm} F_{n+1}(y_k)$ are used to obtain values for the neighboring y_k.

The main advantage

We only need $O(n)$ operations to calculate the integrals

THANK YOU FOR ATTENTION!

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The research was supported by the RFBR (project 18-01-00910)