Numerical methods for computing risk measures in Lévy models

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Outline

1. Risk measures
2. Lévy processes: a short reminder
3. Computing VaR in Lévy models
4. Computing intra-horizon VaR in Lévy models
5. Computing the cost of illiquidity in Lévy models: lookback options
6. Numerical examples

Outline

Historical background

Options

Option (under certain conditions) gives the right to the owner to buy (call) or sell (put) an underlying asset for a fixed price at a fixed date (or during a certain period).

Option types

- European
- American
- Exotics (barrier, lookback, one touch digital, etc)

Historical background

Black-Scholes model

$$
\begin{aligned}
B_t &= B\exp(rt), t \geqslant 0 \\
S_t &= S\exp((r-\sigma^2/2)t + \sigma W_t), t \geqslant 0,
\end{aligned}
$$

B_t – bond price, S_t – stock price, W_t – Brownian motion

Black-Scholes equation

$$
\frac{\partial U}{\partial t}(t,S) + rS\frac{\partial U}{\partial S}(t,S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2}(t,S) - rU(t,S) = 0,
$$

$U(S,t)$ – option price at time t and stock price S.

Main definitions

Risk measures

- The Value-at-Risk (VaR)
- The intra-horizon Value-at-Risk (iVaR)
- The cost of illiquidity

References

BASEL COMMITTEE: Overview of the amendment to the capital accord to incorporate market risks. *Basel Committee on Banking Supervision*, 1996.

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Main definitions

VaR

The Value-at-Risk measures the potential loss in value of a risky asset or portfolio at the end of a specified trading horizon with a given confidence level.

Price level hitting risk and iVaR

Price level hitting risk is a first-passage probability of the underlying stock price hitting the critical stock price level. The intra-horizon Value-at-Risk is the potential value of the loss level exceeded during a fixed time horizon with a given probability.

The cost of illiquidity

According to Longstaff (1995), an expected difference between the maximal stock price over the period and the price in the end of the period gives an upper bound for the value of the stock illiquidity.

Risk measures and option prices

VaR and European digital options

European digital contract pays \$1 at the maturity date T if the underlying asset price S_T crossed a prefixed barrier H, and nothing otherwise.

Price level hitting risk, iVaR and one-touch digital options

A first touch digital contract pays \$1, as a stock price S_t for the first time crosses a prefixed barrier H. If up to the date T the price does not cross the barrier H, the option becomes worthless.

The cost of illiquidity and lookback options

Consider an investor who cannot buy (sell) a stock during a certain time period. A floating strike European lookback call (put) gives the option holder the right to buy (sell) an asset at its lowest (highest) price during the life of the option.

Illiquidity as a lookback option

REPO (repurchase agreement)

In a repo, one party sells an asset (usually fixed-income securities) to another party at one price at the start of the transaction and commits to repurchase the fungible assets from the second party at a different price at a future date or (in the case of an open repo) on demand.

If the seller defaults during the life of the repo, the buyer (as the new owner) can sell the asset to a third party to offset his loss. The asset therefore acts as collateral and mitigates the credit risk that the buyer has on the seller.

During the life of the repo, the asset becomes illiquid for the seller. REPO operations are typically short-term. As an illiquidity risk was estimated before a repo, an investor should dynamically monitor the risk observing the asset prices.

Historical background

Option valuation under Lévy processes has been dealt with by a host of researchers.
However, the pricing options in exponential Lévy models still remains a mathematical and computational challenge.

Methods for pricing path-dependent options

- Monte Carlo methods
- Finite difference schemes
- Integral transform methods
- Wiener-Hopf method

Platform of numerical methods for computational finance

Premia

The program platform Premia (www.premia.fr) developed by the "MathRisk" team at INRIA (the French national institute for research in computer science and control) and financially supported by a consortium of French banks (Credit Agricole Corporate and Investment Bank, Natixis and others).
Premia is a software designed for option pricing, hedging and financial model calibration. It is provided with it's C/C++ source code and an extensive scientific documentation.

PNL

PNL is a numerical library for C and C++ programmers. It is free software under the GNU LGPL. This library is currently used by the PREMIA software. Available at https://github.com/pnlnum/pnl

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KUDRYAVTSEV O., Advantages of the Laplace transform approach in pricing first touch digital options in Levy-driven models, *Boletin de la Sociedad Matematica Mexicana*, 2016

Pricing lookbacks in Lévy models

KUDRYAVTSEV, O. Approximate Wiener–Hopf factorization and Monte Carlo methods for Lévy processes. *Theory Probab. Appl.*, 2019, 64(2).

Outline

Lévy processes: a short reminder

General definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X_t}] = e^{-t\psi(\xi)}$.

The characteristic exponent of Lévy process

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y|\leq 1})\Pi(dy),$$

where σ^2 is the variance of the Gaussian component, and the Lévy measure $\Pi(dy)$ satisfies $\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}\Pi(dy) < +\infty$.
If $\Pi(dx) = \pi(x)dx$, $\pi(x)$ – Lévy density.

Examples of Lévy processes, $\Pi(\mathbf{R}) < \infty$

Jump diffusion

$X_t = \gamma_0 t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$, where W_t – Brownian motion, N_t – Poisson process with intensity λ, and Y_i – i.i.d of jumps.

Kou model

The Lévy density $\pi(x)$, is of the form

$$\pi(x) = (1-p)\lambda\Lambda_- e^{\Lambda_- x}\mathbf{1}_{\{x<0\}} + p\lambda\Lambda_+ e^{-\Lambda_+ x}\mathbf{1}_{\{x>0\}}.$$

where $\Lambda_- > 0, \Lambda_+ > 1$, $0 < p < 1$, $\lambda > 0$.
If we set $c_+ = (1-p)\lambda\Lambda_-$, $c_- = p\lambda\Lambda_+$, $\lambda_+ = \Lambda_-$, $\lambda_- = -\Lambda_+$, then

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where $\sigma > 0$, $\mu = \gamma_0 - \int_{-1}^{1} x\Pi(dx)$, $c_\pm > 0$ and $\lambda_- < -1 < 0 < \lambda_+$.

Examples of Lévy processes, $\Pi(\mathbf{R}) = \infty$

Tempered stable Lévy processes (TSL)

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}],$$

where $\nu_+, \nu_- \in (0,2), \nu_+, \nu_- \neq 1$, $c_+, c_- > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

$$\pi(x) = c_+ e^{\lambda_+ x}|x|^{-\nu_+ - 1}\mathbf{1}_{\{x<0\}} + c_- e^{\lambda_- x}|x|^{-\nu_- - 1}\mathbf{1}_{\{x>0\}}.$$

If $c_- = c_+ = c$ and $\nu_- = \nu_+ = \nu$, then we obtain a KoBoL (CGMY) model.
In the CGMY parametrization $C = c$, $Y = \nu$, $G = \lambda_+$, $M = -\lambda_-$.

Wiener-Hopf factorization

Wiener-Hopf factorization

Let $q>0$, X_t be a Lévy process with characteristic exponent $\psi(\xi)$, $T_q \sim \mathrm{Exp}\, q$, $\bar{X}_t = \sup_{0\leqslant s\leqslant t} X_s$ and $\underline{X}_t = \inf_{0\leqslant s\leqslant t} X_s$ – supremum and infimum processes.

$$\phi_q^+(\xi) = E[e^{i\xi\bar{X}_{T_q}}], \quad \phi_q^-(\xi) = E[e^{i\xi\underline{X}_{T_q}}], \quad \frac{q}{q+\psi(\xi)} = E[e^{i\xi X_{T_q}}].$$

Wiener-Hopf factorization formula reads:

$$\frac{q}{q+\psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi).$$

Useful facts and relations

- $\underline{X}_{T_q}$ and $X_{T_q} - \underline{X}_{T_q}$ – independent;
- $\bar{X}_{T_q}$ and $X_{T_q} - \underline{X}_{T_q}$ – identically distributed.

Expected present value operators:

$$\mathcal{E}_q g(x) = E^x[\int_0^{+\infty} q e^{-qt} g(X_t) dt] = E[g(x + X_{T_q})].$$

$$\mathcal{E}_q^+ g(x) = E^x[\int_0^{+\infty} q e^{-qt} g(\bar{X}_t) dt] = E[g(x + \bar{X}_{T_q})].$$

$$\mathcal{E}_q^- g(x) = E^x[\int_0^{+\infty} q e^{-qt} g(\underline{X}_t) dt] = E[g(x + \underline{X}_{T_q})].$$

$\mathcal{E}_q$ and $\mathcal{E}_q^{\pm}$ as convolution operators

$$\mathcal{E}_q g(x) = \int_{-\infty}^{+\infty} g(x+y) P_q(y) dy, \quad \mathcal{E}_q^{\pm} g(x) = \int_{-\infty}^{+\infty} g(x+y) P_q^{\pm}(y) dy,$$

where $P(y)$, $P_{\pm}(y)$ are probability densities with

$$\mathrm{supp} P_q^+ \subset [0, +\infty), \mathrm{supp} P_q^- \subset (-\infty, 0].$$

Explicit WHF: Gaussian Lévy process

Let $X_t = \gamma_0 t + \sigma W_t$, then

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\gamma\xi.$$

For $q > 0$,the equation $q + \psi(\xi) = 0$ has two roots $-i\beta_-$ and $-i\beta_+$, where $\beta_- < 0$ and $\beta_+ > 0$.

The function $q(q + \psi(\xi))^{-1}$ admits WHF with

$$\phi_q^+(\xi) = \frac{\beta_+}{\beta_+ - i\xi}, \quad \phi_q^-(\xi) = \frac{-\beta_-}{-\beta_- + i\xi}.$$

The functions ϕ_q^- and ϕ_q^+ are chf of exponential distributions on negative ans positive half-lines, respectively:

$$P_q^-(dx) = -\beta_- e^{-\beta_- x}\mathbf{1}_{(-\infty;0]}(x)dx, P_q^+(dx) = \beta_+ e^{-\beta_+ x}\mathbf{1}_{[0;+\infty)}(x)dx.$$

Explicit WHF: Kou model

In Kou model

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}.$$

For $q > 0$,the equation $q + \psi(\xi) = 0$ has four roots $-i\beta_1^-$, $-i\beta_0^-$, $-i\beta_0^+$ and β_1^+, where $\beta_1^- < \lambda_- < \beta_0^- < 0 < \beta_0^+ < \lambda_+ < \beta_1^+$.

The function $q(q + \psi(\xi))^{-1}$ admits WHF with

$$\begin{aligned} \phi_q^+(\xi) &= \frac{\lambda_+ - i\xi}{\lambda_+} \prod_{j=0,1} \frac{\beta_j^+}{\beta_j^+ - i\xi}, \\ \phi_q^-(\xi) &= \frac{-\lambda_- + i\xi}{-\lambda_-} \prod_{j=0,1} \frac{-\beta_j^-}{-\beta_j^- + i\xi}. \end{aligned}$$

Definition

Direct Fourier transform $\mathcal{F}_{x\to\xi}$:

$$\hat{g}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} g(x)dx.$$

Inverse Fourier transform $\mathcal{F}^{-1}_{\xi\to x}$:

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\hat{g}(\xi)d\xi.$$

Some properties

- $\mathcal{F}_{x\to\xi}\mathcal{F}^{-1}_{\xi\to x} = I$ and $\mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{x\to\xi} = I$
- $\mathcal{F}_{x\to\xi}(g * f) = \overline{\mathcal{F}_{x\to\xi}(g)} \cdot \mathcal{F}_{x\to\xi}(f) = \mathcal{F}_{x\to\xi}(g) \cdot \overline{\mathcal{F}_{x\to\xi}(f)}$,
  where $(g * f)(x) = \int_{-\infty}^{+\infty} g(x+y)f(y)dy = \int_{-\infty}^{+\infty} g(z)f(z-x)dz$.

Pseudo-differential operator (PDO)

A PDO $A = a(D)$ with the symbol $a(\xi)$ acts as follows ($D = -i\frac{d}{dx}$):

$$Ag(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(\xi)\hat{g}(\xi)d\xi.$$

In short, $Ag(x) = \mathcal{F}^{-1}_{\xi\to x} a(\xi)\mathcal{F}_{x\to\xi} g(x)$

$\mathcal{E}$ and $\mathcal{E}^{\pm}$ as PDO

$$\mathcal{E}_q g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} q(q+\psi(\xi))^{-1}\hat{g}(\xi)d\xi,$$

$$\mathcal{E}_q^{\pm} g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \phi_q^{\pm}(\xi)\hat{g}(\xi)d\xi.$$

WHF in an operator form: $\mathcal{E} = \mathcal{E}^+\mathcal{E}^- = \mathcal{E}^-\mathcal{E}^+$.

Outline

Computing VaR in Lévy models

The method for computing VaR implemented into Premia 14 is closely related to the papers Kim et al. (2010)

- The numerical method for computing standard risk measures in infinitely divisible distributions is based on the Fourier Transform technique.
- The method efficiently recovers the cumulative distribution function from the characteristic function using the inversion theorem by means of the Fast Fourier Transform algorithm.

Reference

Kim, Y.S., Rachev, S., Bianchi, M.S., Fabozzi, F.J.
Computing VaR and AVar in Infinitely Divisible Distributions.
Probability and Mathematical Statistics, 2010

Computing VaR in Lévy models

Let the random variable X represents the loss of a portfolio. Let $F_X(x) = \mathbf{P}(X < x)$, $p_X(x) = \frac{d}{dx}F_X(x)$, $\phi(\xi) = E[e^{i\xi X}]$ are cdf, pdf and chf of X, respectively.
The VaR of X at tail probability γ is defined as follows.

$$\mathrm{VaR}_\gamma(X) = \inf\{y \in \mathbf{R} | F_X(y) \geqslant \gamma\}.$$

If $F_X(x)$ is continuous, then

$$F_X(x) = E[\mathbf{1}_{X<x}] = \int_{-\infty}^{x} p_X(y)dy.$$

and the following formula is valid:

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\gamma).$$

Computing VaR in Lévy models

If the probability density p_X is known, one can apply a quadrature rule for computing numerically VaR.
If $X = Y_T$, where Y_t – Lévy process, then X – i.d.d. In the general case of i.d.d, p_X can be expressed in terms of chf $\phi_X(\xi)$, by using the inverse Fourier transform of a measure (it differs in sign in front of i from the FT for functions)

$$p_X(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi} \phi_X(\xi) d\xi.$$

One can also express F_X in terms of the Fourier integral

$$F_X(x) = \frac{e^{x\rho}}{\pi} \mathrm{Re} \int_0^{\infty} e^{-ix\xi} \frac{\phi_X(\xi + i\rho)}{\rho - i\xi} d\xi, x \in \mathbf{R},$$

where $\rho > 0$.

Computing VaR in exponential Lévy models

Let the stock price $S_t = S_0 e^{X_t}$ is an exponential Lévy process, then the chf of X_t is given by the formula $\phi_{X_t}(\xi) = e^{-t\psi(\xi)}$. We consider a more general quantity $L_t = S_t - K$, where $K \geqslant 0$.
The computation of the VaR for L_t is straightforward:

$$\mathrm{VaR}_\gamma(L_T) = S_0 e^{\mathrm{VaR}_\gamma(X_T)} - K.$$

Discrete Fourier transform: direct(DFT) and inverse (iDFT)

$$G_l = DFT[g](l) = \sum_{k=0}^{M-1} g_k e^{-2\pi i k l/M}, \quad l = 0, ..., M-1.$$

$$g_k = iDFT[G](k) = \frac{1}{M} \sum_{l=0}^{M-1} G_l e^{2\pi i k l/M}, \quad k = 0, ..., M-1.$$

Computing VaR in exponential Lévy models

Fix the space step $d > 0$ and number of the space points $M = 2^m$. Define the partitions of normalized log-price domain $[-\frac{Md}{2}; \frac{Md}{2})$ by points $x_k = -\frac{Md}{2} + kd$, $k = 0, ..., M-1$, and frequency domain $[-\frac{\pi}{d}; \frac{\pi}{d}]$ by points $\xi_l = -\frac{2\pi l}{dM} + \frac{\pi}{d}$, $l = 0, ..., M$.
For $k = 0, ..., M-1$

$$\begin{aligned} p_X(x_k) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ix_k\xi}\phi_X(\xi)d\xi \\ &\approx \frac{1}{2\pi}\int_{-\pi/d}^{\pi/d} e^{-ix_k\xi}\phi_X(\xi)d\xi \approx \frac{1}{2\pi}\sum_{l=0}^{M-1} e^{-ix_k\xi_l}\phi_X(\xi_l)\frac{2\pi}{dM} \\ &\approx \frac{1}{Md}\sum_{l=0}^{M-1} e^{2\pi ikl/M}\phi_X(\xi_l)(-1)^{l+k} = (-1)^k\frac{1}{d}iDFT[\tilde{\phi_X}](k), \end{aligned}$$

where $(\tilde{\phi_X})_l = \phi_X(\xi_l)\cdot(-1)^l$ and $\phi_X(\xi) = e^{-T\psi(\xi)}$.

Outline

The Fast Wiener-Hopf factorization method (FWHF-method)

In Kudryavtsev (2016) the fast, accurate and universal numerical method for pricing one touch digital options under Lévy models was developed.

- The numerical method is based on an efficient approximation of the Wiener-Hopf factors and the FFT-algorithm.
- In contrast to finite difference methods the FWHF-method deals with the characteristic exponent of the process.
- Acceleration technique based on Laplace transform inversion formulas is applied.

Reference

KUDRYAVTSEV O., Advantages of the Laplace transform approach in pricing first touch digital options in Levy-driven models *Boletin de la Sociedad Matematica Mexicana*, 2016

Laplace transform of cdf functions

WH method

- Denote by $F_+(x,T) = \mathbf{P}(\bar{X}_T < x)$ and $F_-(x,T) = \mathbf{P}(\underline{X}_T < x)$
- Apply Laplace transform to $F_+(-x,T)$, $x<0$:

$$\begin{aligned}\hat{F}_+(-x,q) &= \int_0^{+\infty} e^{-qt} E^x\left[\mathbf{1}_{(-\infty,0)}(\bar{X}_t)\right] dt \\ &= q^{-1} E\left[\mathbf{1}_{(-\infty,0)}(x+\bar{X}_{T_q})\right] = q^{-1}\mathcal{E}_q^+ \mathbf{1}_{(-\infty,0)}(x)\end{aligned}$$

- Apply Laplace transform to $F_-(-x,T)$, $x>0$:

$$\begin{aligned}\hat{F}_-(-x,q) &= \int_0^{+\infty} e^{-qt} E^x\left[\mathbf{1}_{(-\infty,0)}(\underline{X}_t)\right] dt \\ &= q^{-1} E\left[\mathbf{1}_{(-\infty,0)}(x+\underline{X}_{T_q})\right] = q^{-1}\mathcal{E}_q^- \mathbf{1}_{(-\infty,0)}(x)\end{aligned}$$

Numerical Laplace transform inversion: the Gaver-Stehfest algorithm

An approximate formula for $f(\tau)$ can be written as follows

$$f(\tau) \approx \frac{1}{\tau}\sum_{k=1}^{N} \omega_k \cdot \tilde{f}\left(\frac{\alpha_k}{\tau}\right), \qquad 0 < \tau < \infty,$$

$$\begin{aligned} N &= 2n; \\ \alpha_k &= k \ln(2) \\ \omega_k &:= \frac{(-1)^{n+k}\ln(2)}{n!} \sum_{j=[(k+1)/2)]}^{\min\{k,n\}} j^{n+1} C_n^j C_{2j}^j C_j^{k-j}, \end{aligned}$$

where $[x]$ – integer part of x and $C_L^K = \frac{L!}{(L-K)!K!}$ – binomial coefficients.

Numerical method

Key ideas

- Approximate Wiener-Hopf factors $\phi_q^{\pm}(\xi)$.
- Apply the Laplace transform to $F_{\pm}(-x, T)$
- Find at q specified by the Gaver-Stehfest algorithm:

$$\hat{F}_{+}(-x, q) = q^{-1}\mathcal{E}_q^{+}\mathbf{1}_{(-\infty,0)}(x) \quad \hat{F}_{-}(-x, q) = q^{-1}\mathcal{E}_q^{-}\mathbf{1}_{(-\infty,0)}(x).$$

- Cdf $F_{\pm}(x, T)$ can be recovered from $\hat{F}_{\pm}(-x, q)$ by the Gaver-Stehfest algorithm.
- If the cdf F_X is known then one may find $F_X^{-1}(\gamma)$, where γ is a confidence level for iVaR.

Approximate Wiener-Hopf factorization

The Fast Wiener-Hopf factorization method (FWHF-method)

- In Kudryavtsev and Levendorskiĭ (2009) the fast, accurate and universal numerical method for pricing barrier option under Lévy models was developed.
- In Kudryavtsev (2016) the approximate factorization was generalized; convergence of the method was accelerated.

Reference

KUDRYAVTSEV, O.E., AND S.Z. LEVENDORSKIĬ, “Fast and accurate pricing of barrier options under Levy processes”, *J. Finance and Stochastics*, 2009, V. 13, N. 4, 531-562

KUDRYAVTSEV O. Advantages of the Laplace transform approach in pricing first touch digital options in Lévy-driven models. Boletín de la Sociedad Matemática Mexicana, 2016, vol. 22(2), pp. 711–731.

Construction of approximation of factors in the Wiener-Hopf factorization formula

Setup

ν – order of Lévy process
$\psi(\xi)$ – characteristic exponent

$$\begin{aligned} \Lambda_-(\xi) &= \lambda_+^{\nu_+}(\lambda_+ + i\xi)^{-\nu_+}; \\ \Lambda_+(\xi) &= (-\lambda_-)^{\nu_-}(-\lambda_- - i\xi)^{-\nu_-}; \\ \Phi(\xi) &= q\Big((q+\psi(\xi))\Lambda_+(\xi)\Lambda_-(\xi)\Big)^{-1}. \end{aligned}$$

$$\Phi(\xi) = \Phi^+(\xi)\Phi^-(\xi).$$

Construction of approximation of factors in the Wiener-Hopf factorization formula

Explicit formulas for approximations of $\phi^{\pm}$

For small positive d and large $M(=2^n)$, set

$$
\begin{aligned}
b_k^d &= \frac{d}{2\pi}\int_{-\pi/d}^{\pi/d} \ln\Phi(\xi)e^{-i\xi kd}d\xi, \quad k\neq 0,\\
b_{d,M}^{+}(\xi) &= \sum_{k=1}^{M/2} b_k^d(\exp(i\xi kd)-1),\\
b_{d,M}^{-}(\xi) &= \sum_{k=-M/2+1}^{-1} b_k^d(\exp(i\xi kd)-1);\\
\Phi^{\pm}(\xi) &\approx \exp(b_{d,M}^{\pm}(\xi)),\\
\phi_q^{\pm}(\xi) &= \Lambda_{\pm}(\xi)\Phi^{\pm}(\xi).
\end{aligned}
$$

Approximation of $\mathcal{E}, \mathcal{E}^+$ and $\mathcal{E}^-$ using FFT

Approximation of $\mathcal{E}$

Set $p(\xi) = q(q + \psi(\xi))^{-1}$,

$$(\mathcal{E}g)(x_k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix_k\xi} p(\xi)\hat{g}(\xi) d\xi$$

$$(\mathcal{E}g)(x_k) \approx iDFT[p. * DFT[g]](k), \quad k = 0, ..., M - 1.$$

Approximation of $\mathcal{E}^+$ and $\mathcal{E}^-$

$$(\mathcal{E}^+g)(x_k) = iDFT[\phi_q^+ \cdot DFT[g]](k), \quad k = 0, ..., M - 1.$$

$$(\mathcal{E}^-g)(x_k) = iDFT[\phi_q^- \cdot DFT[g]](k), \quad k = 0, ..., M - 1.$$

Outline

The problem for exotic options

We consider options, whose payoff at maturity date T depends on $(X_T, \underline{X}_T)$.

Consider

$$V(T,x) = E^x\left[g(X_T, \underline{X}_T)\right],$$

where
time 0 is the beginning of a period under consideration,
T – the final date,
$g(X_T, \underline{X}_T)$ – the payoff at time T.

Main ideas for deterministic approaches

Carr's randomization or the Laplace transform reduces the pricing problem to the calculation of the appropriate sequence of stationary generalized Black-Scholes equations subject to the correspondent boundary conditions.

Each equation can be solved using the Wiener-Hopf factorization method and the Fast Fourier Transform algorithm when the factors are known.

An efficient approximation of the Wiener-Hopf factors in the exact formula for the solution is obtained by the FWHF-method.

Time randomization and Laplace transform

Laplace transform

$$\begin{aligned}\hat{V}(q,x) &= \int_0^{+\infty} e^{-qt} E^x\left[g(X_t, \underline{X}_t)\right] dt \\ &= E\left[\int_0^{\infty} e^{-qt} g(x+X_t, x+\underline{X}_t) dt\right] \\ &= q^{-1} E[g(x+X_{T_q}, x+\underline{X}_{T_q})] \\ &= q^{-1} E[g(x+\bar{X}_{T_q}+\underline{X}_{T_q}, x+\underline{X}_{T_q})].\end{aligned}$$

$$\begin{aligned}\frac{(-1)^{n-1} q^n}{(n-1)!} \partial_q^{n-1} \hat{V}(q,x) &= \frac{q^n}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-qt} E^x\left[g(X_t, \underline{X}_t)\right] dt \\ &= E[g(x+X_{\Gamma(n,q)}, x+\underline{X}_{\Gamma(n,q)})].\end{aligned}$$

Numerical Laplace transform inversion

Post-Widder formula

If $f(\tau)$ is a function of a nonnegative real variable τ and the Laplace transform $\tilde{f}(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau)\, d\tau$ is known, the approximate Post-Widder formula for $f(\tau)$ can be written as

$$f(\tau) = \lim_{N\to\infty} f_N(\tau); \quad f_N(\tau) := \frac{(-1)^{N-1}}{(N-1)!}\left(\frac{N}{\tau}\right)^N \tilde{f}^{(N-1)}\left(\frac{N}{\tau}\right),$$

where $\tilde{f}^{(N)}(\lambda)$ – Nth derivative of the Laplace transform $\tilde{f}$ at λ. The convergence $f_N(\tau)$ to $f(\tau)$ as $N \to \infty$ is slow (of order N^{-1})

Reference

J. Abate and W. Whitt, A unified framework for numerically inverting Laplace transforms, *INFORMS Journal on Computing*, 2006.

A generalized Wiener-Hopf Monte Carlo method

Key ideas

- Approximate Wiener-Hopf factors $\phi_q^{\pm}(\xi)$ by using the FFT for real-valued functions.
- Apply the Laplace transform to $F_{\pm}(-x, T)$
- Find at q specified by the Gaver-Stehfest algorithm:

$$\hat{F}_{+}(x, q) = q^{-1}\mathcal{E}^{+}\mathbf{1}_{(-\infty,0)}(x) \quad \hat{F}_{-}(x, q) = q^{-1}\mathcal{E}^{-}\mathbf{1}_{(-\infty,0)}(x).$$

- Cdf $F_{\pm}(x)$ can be recovered from $\hat{F}_{\pm}(-x, q)$ by the Gaver-Stehfest algorithm.
- If the cdf F_X is known then one may simulate X by using samples from $F_X^{-1}(U)$, where U is a uniform distribution on $(0, 1)$.

Lookback options: floating strike

European floating strike lookback put

$$V(T,x) = E^x\left[e^{-rT}(Se^{\bar{X}_T} - Se^{X_T})\right],$$

Seasoned European floating strike lookback put

$$V(T_1, T_2; x, h) = E_{T_1}\left[e^{-r(T_2-T_1)}(Se^{\bar{X}_{T_2}} - Se^{X_{T_2}})|X_{T_1} = x, \bar{X}_{T_1} = h\right].$$

Set $T = T_2 - T_1$.

$$\begin{aligned} V(T,x) &= E^x\left[e^{-rT}S(e^{\max\{\bar{X}_T,h\}} - e^{X_T})\right] \\ &= E^x\left[e^{-rT}S(e^{\bar{X}_T} - e^{X_T})\right] + \\ &= E^x\left[e^{-rT}(H - Se^{\bar{X}_T})\mathbf{1}_{\{\bar{X}_T<h\}}\right]. \end{aligned}$$

$H(= Se^h)$ – predefined maximum, $E^x\left[e^{-rT}e^{X_T}\right] = e^x$.

Lookback options: fixed strike

European fixed strike lookback put

$$V(T,x) = E^x\left[e^{-rT}(K - Ke^{\underline{X}_T})_+\right],$$

Seasoned European fixed strike lookback put

$$V(T_1, T_2; x, h) = E_{T_1}\left[e^{-r(T_2-T_1)}(K - Ke^{\underline{X}_{T_2}})_+ | X_{T_1} = x, \underline{X}_{T_1} = h\right].$$

Set $T = T_2 - T_1$.

$$\begin{aligned} V(T,x) &= E^x\left[e^{-rT}(K - Ke^{\min\{\underline{X}_T, h\}})_+\right] \\ &= E^x\left[e^{-rT}(K - Ke^{\underline{X}_T})_+\right] + \\ &= E^x\left[e^{-rT}\left((H - Ke^{\underline{X}_T})_+ - (K - Ke^{\underline{X}_T})_+\right)\mathbf{1}_{\{\underline{X}_T > h\}}\right]. \end{aligned}$$

$H(= Ke^h)$ – predefined minimum

Outline

1. Risk measures
2. Lévy processes: a short reminder
3. Computing VaR in Lévy models
4. Computing intra-horizon VaR in Lévy models
5. Computing the cost of illiquidity in Lévy models: lookback options
6. Numerical examples

Numerical examples

gAWHF&MC-method

The algorithm of the generalized approximate Wiener-Hopf factorization Monte Carlo method was published in Kudryavtsev O.E.(2019). We will refer to it as gAWHF&MC-method.

Bibliography

KUDRYAVTSEV, O.E., "Approximate Wiener–Hopf factorization and Monte Carlo methods for Lévy processes", *Theory Probab. Appl.*, 2019, Vol. 64, No. 2.

Numerical examples

We check the performance of the gAWHF&MC-method against prices obtained by deterministic methods: the FWHF&GS-method from Kudryavtsev O., Levendorskii S. (2011) and the ParaiLT-method from Boyarchenko S. I., Levendorskii S. Z.(2013).

Bibliography

KUDRYAVTSEV, O.E., AND S.Z. LEVENDORSKIǏ, “Efficient pricing options with barrier and lookback features under Levy processes”, Working paper, 2011, 29 pp. Available at SSRN.

BOYARCHENKO S. I., LEVENDORSKII S. Z. “Efficient Laplace inversion, Wiener-Hopf factorization and pricing lookbacks”, *International Journal of Theoretical and Applied Finance*, 2013, vol. 16(3), 1350011.

Experiment setup

We consider European fixed strike lookback put options under the TSL model, and use the same parameters of the KoBoL (CGMY) process as in Boyarchenko S. I., Levendorskii S. Z.(2013):// $c = 0.2395$, $\lambda_+ = 3.0$, $\lambda_- = -10.0$, $\nu = 1.2$ ($C = 0.2395$, $G = 3.0$, $M = 10.0$, $Y = 1.2$ in CGMY parametrization).

The remaining parameters are strike price $K = 100$, the dividend rate $d = 0$ and interest rate $r = 0.04$. The drift parameter μ is fixed by EMM-requirement. We consider 2 maturities $T = 0.1$ (short) and $T = 2$ (long).

The computations performed in 10 points $x_k = \ln(S/K)(= 0.02; 0.04; ...; 0.2)$, where S – initial spot price.

PC characteristics: Intel Core(TM)i5 CPU, 1.7GHz, 4 GB RAM, Windows 7 Professional with 64-bit

Numerical examples

European lookbacks

For verification of the accuracy of our method, we calculate prices for the fixed strike lookback put by using the gAWHF&MC-method, the FWHF&GS-method from the ParaiLT-method.

The prices of the FWHF&GS-method were obtained using the code implemented into the program platform Premia (www.premia.fr).

The prices of the ParaiLT-method were taken from the table 3 Boyarchenko S. I., Levendorskii S. Z.(2013), as well as the benchmarks.

gAWHF&MC-prices converge very fast and agree with the benchmarks. All the methods are in agreement.
gAWHF&MC-method for pricing lookback options could be considered as a competitor to the deterministic methods.

Convergence of gAWHF&MC

Fixed strike lookback put prices and MC-errors. Short maturity

Parameters	$h=0.001,\ N=10^4$		$h=0.001,\ N=10^5$		$h=0.001,\ N=10^6$	
$x=\ln(S/K)$	price	error	price	error	price	error
0.02	5.34748	0.147	5.38840	0.046	5.39141	0.015
0.04	4.21819	0.140	4.25655	0.044	4.26282	0.014
0.06	3.34641	0.132	3.37984	0.042	3.38694	0.013
0.08	2.67288	0.123	2.69821	0.039	2.70609	0.012
...	...	...	...	...	...	...
0.2	0.81726	0.079	0.81431	0.025	0.81986	0.008
Parameters	$h=0.0005,\ N=10^4$		$h=0.0005,\ N=10^5$		$h=0.0005,\ N=10^6$	
$x=\ln(S/K)$	price	error	price	error	price	error
0.02	5.33876	0.147	5.37970	0.046	5.38274	0.015
0.04	4.21152	0.140	4.24982	0.044	4.25610	0.014
0.06	3.34122	0.132	3.37463	0.042	3.38172	0.013
0.08	2.66890	0.123	2.69414	0.039	2.70203	0.012
...	...	...	...	...	...	...
0.2	0.81631	0.079	0.81333	0.025	0.81888	0.008
Parameters	$h=0.0001,\ N=10^4$		$h=0.0001,\ N=10^5$		$h=0.0001,\ N=10^6$	
$x=\ln(S/K)$	price	error	price	error	price	error
0.02	5.33171	0.147	5.37264	0.046	5.37571	0.015
0.04	4.20611	0.140	4.24435	0.044	4.25064	0.014
0.06	3.33699	0.132	3.37037	0.042	3.37745	0.013
0.08	2.66565	0.123	2.69079	0.039	2.69869	0.012
...	...	...	...	...	...	...
0.2	0.81547	0.079	0.81246	0.025	0.81801	0.008

Convergence of gAWHF&MC

Fixed strike lookback put prices and MC-errors. Long maturity

Parameters	$h = 0.001,\ N = 10^5$		$h = 0.001,\ N = 10^6$		$h = 0.001,\ N = 10^7$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	28.29943	0.125	28.27669	0.040	28.26846	0.013
0.04	27.15904	0.126	27.13597	0.040	27.12789	0.013
0.06	26.05742	0.127	26.03453	0.040	26.02668	0.013
0.08	24.99020	0.128	24.96794	0.040	24.96014	0.013
...	...	...	...	...	...	...
0.2	19.22577	0.125	19.21361	0.040	19.20667	0.013
Parameters	$h = 0.0005,\ N = 10^5$		$h = 0.0005,\ N = 10^6$		$h = 0.0005,\ N = 10^7$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	28.29156	0.125	28.26881	0.040	28.26058	0.013
0.04	27.15145	0.126	27.12838	0.040	27.12031	0.013
0.06	26.05008	0.127	26.02719	0.040	26.01934	0.013
0.08	24.98308	0.128	24.96082	0.040	24.95302	0.013
...	...	...	...	...	...	...
0.2	19.21983	0.125	19.20769	0.040	19.20075	0.013
Parameters	$h = 0.0001,\ N = 10^5$		$h = 0.0001,\ N = 10^6$		$h = 0.0001,\ N = 10^7$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	28.28453	0.125	28.26178	0.040	28.25354	0.013
0.04	27.14467	0.126	27.12160	0.040	27.11352	0.013
0.06	26.04351	0.127	26.02061	0.040	26.01277	0.013
0.08	24.97671	0.128	24.95444	0.040	24.94665	0.013
...	...	...	...	...	...	...
0.2	19.21448	0.125	19.20235	0.040	19.19541	0.013

Comparison of gAWHF&MC with deterministic methods

Errors. Short maturity

x	0.02	0.04	0.06	0.08	...	0.20	Time
V_{put}	5.37205	4.24803	3.37586	2.69765	...	0.81512	400-1900
Para iLT							
ϵ=E-01	-0.053	-0.038	-0.028	-0.014	...	0.0064	0.03-0.15
ϵ=E-02	-0.0054	-0.0041	-0.0032	-0.0025	...	-0.0007	0.55-1.49
FWHF& GS_7							
$h = 0.001$	0.00669	0.00633	0.00546	0.00447	...	0.00094	0.078
$h = 0.0005$	0.00352	0.00326	0.00276	0.00223	...	0.00043	0.188
$h = 0.0002$	0.00124	0.00111	0.00091	0.00072	...	0.00009	0.39
gAWHF&MC							
$h = 0.001$							
$N = 10^4$	-0.0246	-0.02984	-0.02945	-0.02478	...	0.00215	0.156
$N = 10^5$	0.0164	0.00852	0.00398	0.00055	...	-0.00081	0.1880
$N = 10^6$	0.0194	0.01479	0.01108	0.00843	...	0.00474	0.797
$h = 0.0005$							
$N = 10^4$	-0.0333	-0.03650	-0.03464	-0.02875	...	0.00120	0.266
$N = 10^5$	0.0077	0.00180	-0.00123	-0.00352	...	-0.00178	0.328
$N = 10^6$	0.0107	0.00808	0.00586	0.00438	...	0.00377	0.906
$h = 0.0001$							
$N = 10^4$	-0.0403	-0.04191	-0.03887	-0.03201	...	0.00036	1.125
$N = 10^5$	0.0006	-0.00367	-0.00548	-0.00686	...	-0.00266	1.172
$N = 10^6$	0.0037	0.00262	0.00160	0.00104	...	0.00289	1.813

Comparison of gAWHF&MC with deterministic methods

Errors. Long maturity

x	0.02	0.04	0.06	0.08	...	0.20	Time
V_{put}	28.25454	27.11439	26.01360	24.94750	...	19.19671	873-2762
Para iLT							
ϵ=E-03	-0.068	-0.062	-0.057	-0.052	...	-0.029	0.23-1.13
ϵ=E-04	-0.0043	-0.0058	-0.0060	-0.0057	...	-0.0033	1.36-5.1
FWHF&GS_7							
$h = 0.001$	0.02686	0.02731	0.02770	0.02790	...	0.0281	0.093
$h = 0.0005$	0.01266	0.01291	0.01300	0.01310	...	0.0132	0.187
$h = 0.0002$	0.00196	0.00221	0.00230	0.00250	...	0.0030	0.359
gAWHF&MC							
$h = 0.001$							
$N = 10^5$	0.0449	0.0446	0.0438	0.0427	...	0.0291	0.218
$N = 10^6$	0.0222	0.0216	0.0209	0.0204	...	0.0169	1.062
$N = 10^7$	0.0139	0.0135	0.0131	0.0126	...	0.0100	7.59
$h = 0.0005$							
$N = 10^5$	0.0370	0.0371	0.0365	0.0356	...	0.0231	0.359
$N = 10^6$	0.0143	0.0140	0.0136	0.0133	...	0.0110	1.062
$N = 10^7$	0.0060	0.0059	0.0057	0.0055	...	0.0040	7.98
$h = 0.0001$							
$N = 10^5$	0.0300	0.0303	0.0299	0.0292	...	0.0178	1.156
$N = 10^6$	0.0072	0.0072	0.0070	0.0069	...	0.0056	1.859
$N = 10^7$	-0.0010	-0.0009	-0.0008	-0.0009	...	-0.0013	9.53