Computing Risk Measures of Variable Annuities in Lévy Models*

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VaR: Main definitions

VaR

The Value-at-Risk measures the potential loss in value of a risky asset or portfolio at the end of a specified trading horizon with a given confidence level.

VaR and European digital options

European digital contract pays $1 at the maturity date T if the underlying asset price S_T crossed a prefixed barrier H, and nothing otherwise.

References

Basel Committee: Overview of the amendment to the capital accord to incorporate market risks. *Basel Committee on Banking Supervision*, 1996.

Kim, Y.S., Rachev, S., Bianchi, M.S., Fabozzi, F.J. Computing VaR and AVar in Infinitely Divisible Distributions. *Probability and Mathematical Statistics*, 2010

Computing VaR in stochastic models

Let the random variable X represents the loss of a portfolio. Let $F_X(x) = \mathrm{P}(X < x)$, $p_X(x) = \frac{d}{dx}F_X(x)$, $\phi(\xi) = E[e^{i\xi X}]$ are cdf, pdf and chf of X, respectively.
The VaR of X at tail probability α is defined as follows.

$$\mathrm{VaR}_\alpha(X) = \inf\{y \in \mathrm{R} | F_X(y) \geqslant \alpha\}.$$

If $F_X(x)$ is continuous, then

$$F_X(x) = E[1_{X<x}] = \int_{-\infty}^{x} p_X(y)dy.$$

and the following formula is valid:

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$

Variable annuity

Definition

A variable annuity is a long-term investment for retirement which benefits offered by insurance companies are usually protected via different mechanisms such as e.g. guaranteed minimum maturity benefits (GMMBs). The computation of the corresponding risk measures is an important issue for the practitioners in risk management.

GMMB

The GMMB riders provide minimum guarantees to protect the investment account of the policyholder. Denoting by τ_x the future lifetime of a policyholder at the age x, the future payment made by the insurer is

$$(G - F_T)^+ 1_{\{\tau_x > T\}}$$

at maturity T for GMMBs, where G is the guarantee level expressed as a percentage of the initial fund value F_0.

Variable annuity

The net liability of GMMBs is
$L_0 := e^{-rT}(G - F_T)^+ 1_{\{\tau_x > T\}} - \int_0^{T\wedge\tau_x} e^{-rs} M_s^e \mathrm{d}s$,
where $F_t := e^{-mt} S_t$ is the current fund value;
$M_t^e := m_e F_t$ is the margin offset income;
S_t is the underlying equity value.

Methods for computing risk measures

The quantile risk measures of the net liabilities of GMMBs have been evaluated by analytical methods, the method based on identity in law, and using conditional moment matching for the Geometric Brownian motion.

The main goal

We propose an efficient numerical method based on the frame projection approach (the PROJ-method) for calculating the Value-at-Risk for variable annuities for a wide class of Lévy processes.

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Lévy processes: a short reminder

General definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X_t}] = e^{-t\psi(\xi)}$.

The characteristic exponent of Lévy process

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y 1_{|y|\leq 1})\Pi(dy),$$

where σ^2 is the variance of the Gaussian component, and the Lévy measure $\Pi(dy)$ satisfies $\int_{\mathrm{R}\setminus\{0\}} \min\{1, y^2\}\Pi(dy) < +\infty$.
If $\Pi(dx) = \pi(x)dx$, $\pi(x)$ – Lévy density.

Examples of Lévy processes, $\Pi(\mathbb{R}) < \infty$

Jump diffusion

$X_t = \gamma_0 t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$, where W_t – Brownian motion, N_t – Poisson process with intensity λ, and Y_i – i.i.d of jumps.

Kou model

The Lévy density $\pi(x)$, is of the form

$$\pi(x) = (1-p)\lambda\Lambda_- e^{\Lambda_- x} 1_{\{x<0\}} + p\lambda\Lambda_+ e^{-\Lambda_+ x} 1_{\{x>0\}}.$$

where $\Lambda_- > 0, \Lambda_+ > 1, 0 < p < 1, \lambda > 0$.
If we set $c_+ = (1-p)\lambda\Lambda_-$, $c_- = p\lambda\Lambda_+$, $\lambda_+ = \Lambda_-$, $\lambda_- = -\Lambda_+$, then

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where $\sigma > 0$, $\mu = \gamma_0 - \int_{-1}^{1} x\Pi(dx)$, $c_\pm > 0$ and $\lambda_- < -1 < 0 < \lambda_+$.

Examples of Lévy processes, $\Pi(\mathbb{R}) = \infty$

Tempered stable Lévy processes (TSL)

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] +$$
$$c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}],$$

where $\nu_+, \nu_- \in (0, 2), \nu_+, \nu_- \neq 1$, $c_+, c_- > 0$, $\mu \in \mathbb{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

$$\pi(x) = c_+ e^{\lambda_+ x}|x|^{-\nu_+-1}1_{\{x<0\}} + c_- e^{\lambda_- x}|x|^{-\nu_- -1}1_{\{x>0\}}.$$

If $c_- = c_+ = c$ and $\nu_- = \nu_+ = \nu$, then we obtain a KoBoL (CGMY) model.
In the CGMY parametrization $C = c$, $Y = \nu$, $G = \lambda_+$, $M = -\lambda_-$.

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The key ideas for the VaR computation

The key quantity for computing $VaR_\alpha(L_0)$

$$\mathrm{P}_0\left(T, G, w\right) = P\left\{e^{-rT}(G - F_T)^+ - \int_0^T e^{-rs} M_s^e ds > w\right\}.$$

To compute $VaR_\alpha(L_0)$ we find a w^* such that ${}_T p_x P_0\left(T, G, w^*\right) = 1 - \alpha$, where ${}_T p_x$ is the probability that a policyholder at age x will survive T units of time, x, $T > 0$.

The problem transformation

$$P_0\left(T, G, w^*\right) = \mathrm{P}\left\{e^{-rT} F_T + m_e \int_0^T e^{-rs} F_s ds < \left(e^{-rT} G - w^*\right)\right\}.$$

Now, we need to find the cdf of $L_0' = e^{-rT} F_T + m_e \int_0^T e^{-rs} F_s ds$.

The frame projection approach (the PROJ-method)

If there is no an explicit formula for the pdf p_{X_T}, it can be recovered by inverting the chf $\phi_{X_T}(\xi)$ using the inverse Fourier transform: $p_{X_T}(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi}\phi_{X_T}(\xi)d\xi$.

In series of papers the frame projection approach (PROJ) was developed. In particular, in Kirkby (2016) the approach was applied for robust pricing Asian under exponential Lévy models.

Coefficient functionals of the orthogonally projected transition density are given by its convolution with a dual B-spline scaling function of the second order, using the characteristic function of the underlying asset.

Reference

Kirkby, J. L. An efficient transform method for Asian option pricing. *SIAM Journal on Financial Mathematics*, 2016.

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The frame projection

The B-spline bases of order p are of particular interest and can be derived as follows. Starting with the Haar scaling function defined by $\varphi^{[0]}(y) := 1_{[-\frac{1}{2},\frac{1}{2}]}(y)$, the p-th order B-spline scaling functions are derived successively by the convolution

$$\varphi^{[p]}(x) = \varphi^{[0]} \star \varphi^{[p-1]}(x) = \int_{-\infty}^{\infty} \varphi^{[p-1]}(y-x) 1_{[-\frac{1}{2},\frac{1}{2}]}(y) dy.$$

Denote by $\phi(\nu)$ a symmetric generator of the B-spline basis. For a fixed $a > 0$, we consider a space of compactly supported basis elements

$$\phi_{a,k}(\nu) := a^{1/2}\phi(a(\nu - \nu_k)),$$

where $\nu_k = (1 - N/2)\Delta x + k\Delta x$, $k = 0, \ldots, N-1$, with $\Delta x = 1/a$. Let $\widetilde{\phi}_{a,k}$ be the dual basis with a generator $\widetilde{\phi}$.

The frame projection

Frames

A frame of a Hilbert space $\mathcal{H}$ is a collection of elements $\{f_k\}_{k\in I\subset Z}$ from $\mathcal{H}$ that spans $\mathcal{H}$. Formally, a collection of elements $\{f_k\}_{k\in I}$ from $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist two constants $0 < A \leqslant B < +\infty$ for which

$$\forall f \in \mathcal{H}, \quad A \leqslant \sum_{k\in I} |(f, f_k)|^2 \leqslant B||f||, (*)$$

In our case, it can be shown that the bounds in (*) are equivalent to

$$A \leqslant \Phi(\xi) \leqslant B, \xi \in [0, 2\pi], \text{where } \Phi(\xi) := \sum_{k\in Z} |\hat{\phi}(\xi + 2\pi k)|^2.$$

Then we have,

$$\hat{\tilde{\phi}}(\xi) = \frac{\hat{\phi}(\xi)}{\Phi(\xi)}.$$

The frame projection

For a fixed resolution $a > 0$, and the generator $\phi(\nu)$, we obtain

$$p_{\Delta\tau}(\nu) \approx \sum_{k=0}^{N-1}\Big(\int_{-\infty}^{+\infty} p_{\Delta\tau}(y)\widetilde{\phi}_{a,k}(y)dy\Big)\phi_{a,k}(\nu)$$

which provides the L_2 projection restricted to $\{\phi_{a,k}(\nu)\}_{k=0}^{k=N-1}$.

One can show that for $\phi(\nu) = (1 - |\nu|)1_{[-1,1]}$,

$$\Phi(\xi) = \frac{1}{3}(2 + \cos(\xi)), \quad \hat{\phi}(\xi) = \Big(\frac{\sin(\xi/2)}{(\xi/2)}\Big)^2.$$

Then we have, $\widehat{\widetilde{\phi}}_{a,k}(\xi) = a^{-1/2}\exp(-i\nu_k\xi)\widehat{\widetilde{\phi}}(\xi/a)$, where $\widehat{\widetilde{\phi}}(\xi) = \frac{12\sin^2(\xi/2)}{\xi^2(2+\cos(\xi/2))}$.

Recall that for general Lévy models, the chf $\phi_{X_t}(\xi) = e^{-t\psi(\xi)}$. Then in general case, $p_{X_{\Delta\tau}}$ can be expressed as follows

$$p_{\Delta\tau}(\nu) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-i\nu\xi - \Delta\tau\psi(\xi)} d\xi.$$

Using the Fourier transform technique, we obtain

$$\begin{aligned} p_{\Delta\tau}(\nu) &\approx \sum_{k=0}^{N-1} \frac{1}{2\pi} \left(\int_{-\infty}^{+\infty} \phi_{X_{\Delta\tau}}(\xi) \widehat{\overline{\phi}}_{a,k}(\xi) d\xi \right) \phi_{a,k}(\nu) \\ &\approx \sum_{k=0}^{N-1} \frac{a^{-1/2}}{\pi} \mathrm{Re} \left(\int_{0}^{+\infty} e^{-i\nu_k\xi - \Delta\tau\psi(\xi)} \widehat{\overline{\phi}}(\xi/a) d\xi \right) \phi_{a,k}(\nu) \\ &\approx \frac{24a^{5/2}}{N} \sum_{k=0}^{N-1} \beta_{a,k} \phi_{a,k}(\nu), \end{aligned}$$

Approximation of L'_0

We set $F'_t = e^{-rt}F_t$ and rewrite L'_0 as $L'_0 = F'_T + m_e \int_0^T F'_s ds$.
Let F'_t be a Lévy process with the characteristic exponent ψ.

Time discretization

Select the following partitions of $[0, T]$:
$\mathrm{T} = \{t_0, t_1, \ldots, t_{NN}\}$, where $t_j = j\Delta t = j\frac{T}{NN}$,
$\mathrm{T}^\star = \{t_0^\star, t_1^\star, \ldots, t_{NN-1}^\star\}$, where $t_j < t_j^\star < t_{j+1}, j = 0, \ldots, NN-1$.
We choose the points $t_j^\star$ in such way that

$$E\left[\int_{t_{j-1}}^{t_j} F'_s ds - \int_{t_{j-1}}^{t_j} F'_{t_j^\star} ds\right] = 0.$$

Since $E[F'_s] = \exp(-s\psi(-i))]$, we obtain the equation for $t_j^\star$:
$\int_{t_{j-1}}^{t_j} \exp(-s\psi(-i)]ds - \int_{t_{j-1}}^{t_j} \exp(-t_j^\star\psi(-i))ds = 0.$

Approximation of L_0'

If $\psi(-i) = 0$, then $t_j^\star$ can be set to any value of the interval $[t_{j-1}, t_j]$. Otherwise, we have

$$t_j^\star = t_{j-1} + \frac{1}{-\psi(-i)} \log\left(\frac{1 - \exp(-\Delta t \psi(-i))}{\psi(-i)\Delta t}\right),$$

Estimating the integral $\int_0^T F_s' ds$ with the Riemann sum, we may write the approximation L_{NN}^ω for L_0' as $L_{NN}^\omega = \sum_{j=0}^{NN-1} \omega_j F_{t_j^\star}' + \omega_{NN} F_{t_{NN}}'$, where $\omega_{NN} = 1$ and the weights $\omega_j = m_e \cdot \Delta t$, $j = 0, \ldots, NN-1$.

If the density of L_{NN}^ω is known, say $p_{L_{NN}^\omega}$, then

$$P_0\left(T, G, VaR_\alpha\left(L_0\right)\right) \approx \int_0^W p_{L_{NN}^\omega}(u) du,$$

where $W = e^{-rT} G - VaR_\alpha\left(L_0\right)$.

Iterative scheme

Proposition

Fix a set of positive weights $\omega = \{\omega_j\}_{j=0}^{NN}$, *and define* $X_j := \frac{\omega_j}{\omega_{j-1}} \exp(R_j)$, $j = 1, \ldots, NN$, *where* $R_j = \log(F_{t_j}/F_{t_{j-1}})$.
Set $\omega'_0 = \log(\omega_{NN}/\omega_{NN-1})$ *and* $Y_0 = \log(X_{NN}) = \omega'_0 + R_{NN}$, *and define* $Y_j = \omega'_j + R_{NN-j} + Z_{j-1}, \quad j = 1, \ldots, NN-1$, $Y_{NN} = R_0 + Z_{NN-1}$,
where $Z_j := \log(1 + \exp(Y_j))$, *and* $\omega'_j := \log\left(\frac{\omega_{NN-j}}{\omega_{NN-(j+1)}}\right)$.
Then

$$L^{\omega}_{NN} \equiv \omega_0 F_0 \exp(Y_{NN}).$$

Notice that $\phi_{Z_j}(\xi) = \int_{\mathbb{R}} (e^y + 1)^{i\xi} p_{Y_j}(y) dy$.
Set $y^* = \log\left(W/(F_0 \cdot \omega_0)\right)$. According the Proposition,

$$P_0\left(T, G, VaR_\alpha\left(L_0\right)\right) \approx \int_{-\infty}^{y^*} p_{Y_{NN}}(y) dy.$$

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Numerical examples. Geometrical Brownian motion model

As a basic example, we consider the GMMBs with the probability ${}_Tp_x = 0.757$ and the risk tolerance levels $\alpha = 0.9$ and $\alpha = 0.95$.

Model parameters

We take the GBM model with the parameters $\sigma = 0.3$, $\mu = 0.09$.

Variable annuity parameters

the instantaneous interest rate: $r = 0.04$,
time to expiry: $T = 10$ years,
the guarantee level: $G_0 = 100$,
the initial fund value $F_0 = 100$,
the annualized mortality rate: $m = 0.01$,
the GMMB coefficient $m_e = 0.003$.

Performance of the PROJ-method

Value-at-Risk in the Geometrical Brownian motion model

	FL[a]	PROJ[b]	PROJ	PROJ
M		10^5	10^6	10^7
N		30	30	30
$VaR_{90\%}$	12.550369	12.545053	12.549986	12.549898
$VaR_{95\%}$	28.935733	28.824677	28.935175	28.935238
time		0.007 s	0.023 s	0.053 s

[a] The FV-method

[b] The PROJ-method: M – the number of space points, N – the number of time steps.

We used $VaR_\alpha(L_0)$ obtained in Feng and Volkmer (2012) as the benchmark.

Numerical examples. The CGMY model

As a basic example, we consider the GMMBs with the probability $_Tp_x = 0.757$ and the risk tolerance level $\alpha = 0.8$.

Model parameters

We take the CGMY model with the parameters $C = 0.0367$, $G = 0.0765$, $M = 7.5515$, $Y = 1.2945$, $\mu = 0.09$.

Variable annuity parameters

the instantaneous interest rate: $r = 0.04$,
time to expiry: $T = 10$ years,
the guarantee level: $G_0 = 75$,
the initial fund value $F_0 = 100$,
the annualized mortality rate: $m = 0.01$,
the GMMB coefficient $m_e = 0.003$.

We use $P_0\left(T, G, VaR_\alpha\left(L_0\right)\right) = \frac{1-\alpha}{_Tp_x} = 0.2642$ as the benchmark.

Numerical examples

We checked the performance of the PROJ-method against the probability $P_0(T, G, VaR_\alpha(L_0))$ obtained by a Monte Carlo method (MC-method) with $VaR_\alpha(L_0)$ obtained by the PROJ-method.

The computations of the $VaR_\alpha(L_0)$ by the PROJ-method performed for $N = 30$ time steps and the number of space points $M(= 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10})$. $VaR_\alpha(L_0)$ converged rapidly to the value 37.10784 (in the partial of second).

We calculated the frequency of crossing the level $W = \exp(-rT)G_0 - 37.10784$ by L_0 with the MC-method with 100,000 sample paths simulations and 4000 time steps along each trajectory.
The benchmark is in 95%-confidence interval $(0.263122, 0.268598)$, the sample mean 0.26586 departs from the benchmark less than 1%.

Conclusion

- In the proposed approach, the probability density of the net liabilities is approximated using the theory of frames and Riesz bases.
- The key element of the numerical method is a new algorithm for calculating the integral of the exponential Levy process, approximated by a discrete sum whose expectation coincides with the expected value of the desired integral.
- Numerical experiments on the application of the developed method for the GBM and CGMY models clearly demonstrate its high accuracy and speed.
- The PROJ method can be applied for calculating expectations when the characteristic function of log return is known.