Pricing double barrier options under Lévy processes with jumps of unbounded variation*

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Pricing options under Lévy processes

Option valuation under Lévy processes has been dealt with by a host of researchers.
However, pricing path-dependent options in exponential Lévy models still remains a computational challenge.

Path-dependent options: barrier options

A double barrier option is a contract which pays the specified amount $G(S_T)$ at the terminal date T, provided during the lifetime of the contract, the price of the stock does not cross specified constant barriers D from above and U from below. When at least one of the barriers is crossed, the option expires worthless or the option owner is entitled to some *rebate*.
If $U = +\infty$, we obtain a down-and-out single barrier option.
If $D = -\infty$, we deal with an up-and-out single barrier option.

Historical background

Methods for pricing barrier options: drawbacks

- Monte Carlo methods: *slow*
- Finite difference schemes: *application entails a detailed analysis of the underlying Lévy process*
- Wiener-Hopf factorization methods:
  - Single barriers: *non trivial approximate formulas are needed in general case*
  - Double barriers: *application involves an iterative solution to the pair of coupled WH-integral equations or matrix factorization*

The main goal

The goal of the current paper is to suggest a new, easy, and effective method to price double barrier options under pure non-Gaussian Lévy processes with jumps of unbounded variation. The main advantage of the approach is applying semi-explicit Wiener-Hopf factorization formulas

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2. Lévy processes: a short reminder
3. Wiener-Hopf factorization
4. The problem setup and general steps
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Lévy processes: a short reminder

General definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$.

The characteristic exponent of a pure non-Gaussian Lévy process

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = -i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y|\le 1}) F(dy),$$

where the Lévy measure $F(dy)$ satisfies $\int_{\mathbb{R}\setminus\{0\}} \min\{1, y^2\} F(dy) < +\infty$. If $F(dx) = \pi(x)dx$, $\pi(x)$ – Lévy density.

Examples of Lévy processes, $F(\mathbb{R}) = \infty$

A Lévy process of unbounded variation

The condition $\int_{\mathbb{R}\setminus\{0\}} \min\{1, |y|\} F(dy) < +\infty$ does not hold.

Tempered stable Lévy processes (TSL) of unbounded variation

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] +$$
$$c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}],$$

where $\nu_+, \nu_- \in (1,2)$, $c_+, c_- > 0$, $\mu \in \mathbb{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. If $c_- = c_+ = c$ and $\nu_- = \nu_+ = \nu$, then we obtain a KoBoL (CGMY) model.

$$\pi(x) = c_+ e^{\lambda_+ x} |x|^{-\nu_+ - 1} 1_{\{x<0\}} + c_- e^{\lambda_- x} |x|^{-\nu_- - 1} 1_{\{x>0\}}.$$

In the CGMY parametrization: $C = c$, $Y = \nu$, $G = \lambda_+$, $M = -\lambda_-$.

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Definition

Direct Fourier transform $\mathcal{F}_{x\to\xi}$:

$$\hat{g}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} g(x)dx.$$

Inverse Fourier transform $\mathcal{F}^{-1}_{\xi\to x}$:

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi} \hat{g}(\xi)d\xi.$$

Some properties

- $\mathcal{F}_{x\to\xi}\mathcal{F}^{-1}_{\xi\to x} = I$ and $\mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{x\to\xi} = I$
- $\mathcal{F}_{x\to\xi}(g * f) = \overline{\mathcal{F}_{x\to\xi}(g)} \cdot \mathcal{F}_{x\to\xi}(f) = \mathcal{F}_{x\to\xi}(g) \cdot \overline{\mathcal{F}_{x\to\xi}(f)}$, where $(g * f)(x) = \int_{-\infty}^{+\infty} g(x+y)f(y)dy = \int_{-\infty}^{+\infty} g(z)f(z-x)dz$.

Wiener-Hopf factorization for $E[e^{i\xi X_{T_q}}]$

Let $q > 0$, X_t be a Lévy process with characteristic exponent $\psi(\xi)$, $T_q \sim \operatorname{Exp} q$, $\bar{X}_t = \sup_{0\leqslant s\leqslant t} X_s$ and $\underline{X}_t = \inf_{0\leqslant s\leqslant t} X_s$ – supremum and infimum processes.

$$\phi_q^+(\xi) = E[e^{i\xi \bar{X}_{T_q}}], \quad \phi_q^-(\xi) = E[e^{i\xi \underline{X}_{T_q}}], \quad \frac{q}{q+\psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi).$$

Wiener-Hopf factorization for operators: $\mathcal{E} = \mathcal{E}^+\mathcal{E}^- = \mathcal{E}^-\mathcal{E}^+$

$$\mathcal{E}_q g(x) = E[g(x + X_{T_q})] = \int_{-\infty}^{+\infty} g(x+y)P(y)dy,$$

$$\mathcal{E}_q^+ g(x) = E[g(x + \bar{X}_{T_q})] = \int_{-\infty}^{+\infty} g(x+y)P_+(y)dy,$$

$$\mathcal{E}_q^- g(x) = E[g(x + \underline{X}_{T_q})] = \int_{-\infty}^{+\infty} g(x+y)P_-(y)(dy),$$

where $P(y)$, $P_\pm(y)$ are probability densities with $P_\pm(y) = 0, \forall \pm y < 0$.

Pseudo-differential operator (PDO)

A PDO $A = a(D)$ with the symbol $a(\xi)$ acts as follows ($D = -i\frac{d}{dx}$):

$$Ag(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(\xi)\hat{g}(\xi) d\xi.$$

In short, $Ag(x) = \mathcal{F}^{-1}_{\xi\to x} a(\xi)\mathcal{F}_{x\to\xi} g(x)$

$\mathcal{E}$ and $\mathcal{E}^{\pm}$ as PDO

$$\mathcal{E}_q g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} q(q+\psi(\xi))^{-1} \hat{g}(\xi) d\xi,$$

$$\mathcal{E}^{\pm}_q g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \phi^{\pm}_q(\xi)\hat{g}(\xi) d\xi.$$

WHF in an operator form: $\mathcal{E} = \mathcal{E}^{+}\mathcal{E}^{-} = \mathcal{E}^{-}\mathcal{E}^{+}$.

Useful facts and relations

Let X_t – Lévy process, and $T_q \sim \mathrm{Exp}\, q$. Then

- $\underline{X}_{T_q}$ and $X_{T_q} - \underline{X}_{T_q}$ – independent;
- $\bar{X}_{T_q}$ and $X_{T_q} - \underline{X}_{T_q}$ – identically distributed.

A semi-explicit factorization: $F((0, +\infty)) = 0$

Denote by $\beta_q^+ = \sup\{\beta \geqslant 0 : q + \psi(-i\beta) = 0\}$, and we obtain:

$$\phi_q^+(\xi) = \frac{\beta_q^+}{\beta_q^+ - i\xi}, \quad \phi_q^-(\xi) = \frac{q(\beta_q^+ - i\xi)}{(q + \psi(\xi))\beta_q^+}.$$

A semi-explicit factorization: $F((-\infty, 0)) = 0$

Denote by $\beta_q^- = \inf\{\beta \leqslant 0 : q + \psi(-i\beta) = 0\}$, and we obtain:

$$\phi_q^-(\xi) = \frac{-\beta_q^-}{-\beta_q^- + i\xi}, \quad \phi_q^-(\xi) = \frac{q(-\beta_q^- + i\xi)}{(q + \psi(\xi))(-\beta_q^-)}.$$

Explicit WHF: Kou model

In Kou model

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}.$$

For $q > 0$,the equation $q + \psi(\xi) = 0$ has four roots $-i\beta_1^-$, $-i\beta_0^-$, $-i\beta_0^+$ and β_1^+, where $\beta_1^- < \lambda_- < \beta_0^- < 0 < \beta_0^+ < \lambda_+ < \beta_1^+$.

The function $q(q+\psi(\xi))^{-1}$ admits WHF with

$$\phi_q^+(\xi) = \frac{\lambda_+ - i\xi}{\lambda_+} \prod_{j=0,1} \frac{\beta_j^+}{\beta_j^+ - i\xi},$$

$$\phi_q^-(\xi) = \frac{-\lambda_- + i\xi}{-\lambda_-} \prod_{j=0,1} \frac{-\beta_j^-}{-\beta_j^- + i\xi}.$$

Approximate Wiener-Hopf factorization

Formulas for WH-factors

For a wide class of Lévy processes X_t, the following integral representations for $\phi_q^+(\xi)$, $\phi_q^-(\xi)$ are valid (see details in Boyarchenko and Levendorskii (2002)):

$$\phi_q^+(\xi) = \exp\left[(2\pi i)^{-1}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-}\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)}d\eta\right];$$

$$\phi_q^-(\xi) = \exp\left[-(2\pi i)^{-1}\int_{-\infty+i\omega_+}^{+\infty+i\omega_+}\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)}d\eta\right],$$

where $\omega_- < 0 < \omega_+$ with ω_-, ω_+ depending on the Lévy process X_t parameters.

The direct computation of $\phi_q^+(\xi)$ and $\phi_q^-(\xi)$ require $O(NM)$ operations, where N is a number of ξ-points and M is a number of points for numerical integration

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The problem for double barrier options

Let T, K, D, U be the maturity, strike, the lower barrier, and the upper barrier, and the stock price $S_t = De^{X_t}$ be an exponential Lévy process under a chosen risk-neutral measure ($\psi(-i) + r = 0$) which has no diffusion component ($\sigma = 0$) and only jumps of finite variation.

Let us introduce $h = \ln U/D$. We consider options, whose payoff at maturity date T depends on $(X_T, \underline{X}_T, \bar{X}_T)$.

Consider

$$V(T,x) = E^x \left[e^{-rT} 1_{\underline{X}_T > 0} 1_{\bar{X}_T < h} G(X_T) \right],$$

where
time 0 is the beginning of a period under consideration,
T – the final date,
0 and h are the absorbing barriers,
$G(X_T)$ – the payoff function at time T provided the barriers has't been crossed.

Time randomization and Laplace transform

$$\hat{V}(q,x) = \int_0^{+\infty} e^{-qt} E^x\left[e^{-rt} G(X_t) 1_{\underline{X}_t>0} 1_{\bar{X}_t<h}\right] dt$$
$$= E^x \left[\int_0^{+\infty} e^{-(q+r)t} G(X_t) 1_{\underline{X}_t>0} 1_{\bar{X}_t<h} dt\right].$$

$$v_n(q,x) := \frac{(-1)^{n-1} q^n}{(n-1)!} \partial_q^{n-1} \hat{V}(q,x)$$
$$= \int_0^{+\infty} \frac{q^n t^{n-1}}{(n-1)!} e^{-(q+r)t} E^x\left[G(X_t) 1_{\underline{X}_t>0} 1_{\bar{X}_t<h}\right] dt$$
$$= E^x \left[\frac{G(X_{\mathrm{T}(n,q+r)})}{(1+r/q)^n} 1_{\underline{X}_{\mathrm{T}(n,q+r)}>0} 1_{\bar{X}_{\mathrm{T}(n,q+r)}<h}\right],$$

$\mathrm{T}(n,q)$ is a Gamma random variable.

Numerical Laplace transform inversion

Post-Widder formula

If $f(\tau)$ is a function of a nonnegative real variable τ and the Laplace transform $\tilde{f}(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau)\, d\tau$ is known, the approximate Post-Widder formula for $f(\tau)$ can be written as

$$f(\tau) = \lim_{N\to\infty} f_N(\tau); \quad f_N(\tau) := \frac{(-1)^{N-1}}{(N-1)!} \left(\frac{N}{\tau}\right)^N \tilde{f}^{(N-1)}\left(\frac{N}{\tau}\right),$$

where $\tilde{f}^{(N)}(\lambda)$ – Nth derivative of the Laplace transform $\tilde{f}$ at λ.
Set $q = T/N$. The convergence $v_N(T/N, x)$ to $V(T, x)$ as $N \to \infty$ is of order N^{-1}.

Single barrier options in Lévy models, Post-Widder formula

Kudryavtsev, O., "An efficient numerical method to solve a special class of integro-differential equations relating to the Levy models" // *Mathematical Models and Computer Simulations*, 2011, V.3., N.6.

Iterative scheme

Taking into account that $\mathrm{T}(n, q+r) \sim \mathrm{T}(n-1, q+r) + T_{q+r}$, we can represent $X_{\mathrm{T}(n,q+r)} = X_{T^1_{q+r}} + \ldots + X_{T^n_{q+r}}$, where $T^1_{q+r}, \ldots, T^n_{q+r}$ are consecutive time increments being independent exponentially distributed random variables with the parameter $q+r$.

Using the relations

$$1_{\underline{X}_{\mathrm{T}(n,q+r)}>0} = 1_{\underline{X}_{\mathrm{T}(n-1,q+r)}>0} 1_{X_{\mathrm{T}(n-1,q+r)}+\underline{X}_{T^n_{q+r}}>0},$$

$$1_{\bar{X}_{\mathrm{T}(n,q+r)}<h} = 1_{\bar{X}_{\mathrm{T}(n-1,q+r)}<h} 1_{X_{\mathrm{T}(n-1,q+r)}+\bar{X}_{T^n_{q+r}}<h},$$

we obtain that for $n = 1, 2, \ldots$

$$v_n(q, x) = E^x \left[\frac{v_{n-1}(q, X_{T_{q+r}})}{(1 + r/q)} 1_{\underline{X}_{T_{q+r}}>0} 1_{\bar{X}_{T_{q+r}}<h} \right],$$

where $v_0(q, x) = G(x) 1_{(0,h)}(x)$.

Convergence

Theorem 1

Let N be a suffciently large natural number. Set $q = T/N$, $v_0(q,x) = G(x)1_{(0,h)}(x)$, and for $n = 1, 2, \ldots$ define

$$v_n(q,x) = E^x\left[\frac{v_{n-1}(q, X_{T_{q+r}})}{(1+r/q)} 1_{\underline{X}_{T_{q+r}}>0} 1_{\bar{X}_{T_{q+r}}<h}\right],$$

where the random time $T_{q+r} \sim \mathrm{Exp}\,(q+r)$.
For a fixed x, $v_N(T/N, x)$ converges to $V(T,x)$ as $N \to \infty$.

Equivalent problems for IDE

$$\begin{aligned}
\frac{1}{q+r}(q+r+\psi(D_x))v_n(x) &= \frac{q}{q+r}v_{n-1}(x), \quad 0 < x < h, \\
v_n(x) &= 0, \quad x \notin (0,h).
\end{aligned}$$

The state-of-art implementation of the Wiener-Hopf method

Pricing single barrier options

One needs to factorize $(q+r)/(q+r+\psi(\xi))$.Then one can calculate the sequence $v_n(q,x)$ with $q = N/T$ and $h = +\infty$ as follows: for $n = 1, \dots, N$

$$\begin{aligned} v_n(q,x) &= \frac{1}{(1+r/q)} E^x[v_{n-1}(q,(X_{T_{q+r}} - \underline{X}_{T_{q+r}}) + \underline{X}_{T_{q+r}}) 1_{\underline{X}_{T_{q+r}}>0}] \\ &= \frac{1}{(1+r/q)} \mathcal{E}^-_{q+r} 1_{(0,+\infty)} \mathcal{E}^+_{q+r} v_{n-1}(q,x). \end{aligned}$$

Unfortunately, in the case of double barrier options (with $h < +\infty$), such formulas are not available for general Lévy models. Instead one needs to factorize the following matrix:

$$\begin{pmatrix} \exp(i\xi h) & 0 \\ (q+r)/(q+r+\psi(\xi)) & \exp(-i\xi h) \end{pmatrix}$$

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Splitting

The key idea behind the method is to represent the process X_t as the sum of spectrally positive jumps X_t^+ with a non-negative drift and spectrally negative jumps X_t^- with a non-positive drift: $X_t = X_t^+ + X_t^-$.

The first step

- Denote by $\psi_+(\xi)$ and $\psi_-(\xi)$ the characteristic exponents of X_t^+ and X_t^-, respectively.
- If the drift $\mu \geqslant 0$:
  $\psi_+(\xi) = -i\mu\xi + \int_0^{+\infty}(1 - e^{i\xi y} + i\xi y 1_{|y|\leq 1})F(dy)$,
  $\psi_-(\xi) = \int_{-\infty}^{0}(1 - e^{i\xi y} + i\xi y 1_{|y|\leq 1})F(dy)$.
- If the drift $\mu < 0$:
  $\psi_+(\xi) = \int_0^{+\infty}(1 - e^{i\xi y} + i\xi y 1_{|y|\leq 1})F(dy)$,
  $\psi_-(\xi) = -i\mu\xi + \int_{-\infty}^{0}(1 - e^{i\xi y} + i\xi y 1_{|y|\leq 1})F(dy)$.

Approximation. Key ideas

Introduce $X_t^{+,i} \sim X_t^+, = 1, ..., 4$ and $X_t^{-,j} \sim X_t^-, j = 5, 6$.
$X_t \sim Y_t = (\underline{X}_{t/2}^{+,1} + \bar{X}_{t/2}^{+,2} + \bar{X}_t^{-,5} + \underline{X}_t^{-,6} + \underline{X}_{t/2}^{+,3} + \bar{X}_{t/2}^{+,4})$.

For a fixed large parameter $q > 0$, we introduce the following approximate process Y_j^q in discrete time:

- $Y_0^q = 0$;
- $Y_1^q = Y_0^q + \underline{X}_{T_q^1/2}^{+,1}$ – a downward movement;
- $Y_2^q = Y_1^q + \bar{X}_{T_q^1/2}^{+,2}$ – an upward movement;
- $Y_3^q = Y_2^q + \bar{X}_{T_q^2}^{-,1}$ – an upward movement;
- $Y_4^q = Y_3^q + \underline{X}_{T_q^2}^{-,2}$ – a downward movement;
- $Y_5^q = Y_4^q + \underline{X}_{T_q^3/2}^{+,3}$ – a downward movement;
- $Y_6^q = Y_5^q + \bar{X}_{T_q^3/2}^{+,4}$ – an upward movement.

Approximation. Key ideas

Let a natural number N be sufficiently large and $q = N/T$.
We approximate $X_{T_{q+r}}$ with Y_6^{q+r}, since the randomized time T_{q+r} converges in quadratic mean to 0 as $N \to +\infty$.

Theorem 2

Let $q > 0$ be sufficiently large. Then for a fixed $\xi \in \mathrm{R}$

$$E[e^{i\xi X(T_q)}] - E[e^{i\xi Y_6^q}] \sim O(q^{-2}) \text{ as } q \to +\infty.$$

For a sufficiently large $q > 0$, we may approximate the values of $\bar{X}_{T_q}$ and $\underline{X}_{T_q}$ by the quantities

$$\begin{aligned}
\underline{Y}_6^q &= \min\{Y_0^q + \underline{X}^{+,1}_{T_q^1/2}, Y_3^q + \underline{X}^{-,6}_{T_q^2}, Y_4^q + \underline{X}^{+,3}_{T_q^3/2}\},\\
\overline{Y}_6^q &= \max\{Y_1^q + \bar{X}^{+,2}_{T_q^1/2}, Y_2^q + \bar{X}^{-,1}_{T_q^2}, Y_5^q + \bar{X}^{+,4}_{T_q^3/2}\}.
\end{aligned}$$

Iterative scheme

$$v_n(x) = E\left[\frac{q}{q+r} v_{n-1}\left(x + Y_6^{q+r}\right) 1_{(0,h)}(x + \underline{Y}_6^{q+r}) 1_{(0,h)}(x + \overline{Y}_6^{q+r})\right]$$

Theorem 3

Let a natural number N be sufficiently large and $q = N/T$. Introduce the following operators:

$$\begin{aligned} \mathcal{E}_+^+ u(x) &= E[u(x + \bar{X}^+_{T_{q+r/2}})],\ \mathcal{E}_-^+ u(x) = E[u(x + \bar{X}^-_{T_{q+r}})]; \\ \mathcal{E}_+^- u(x) &= E[u(x + \underline{X}^+_{T_{q+r/2}})],\ \mathcal{E}_-^- u(x) = E[u(x + \underline{X}^-_{T_{q+r}})]. \end{aligned}$$

One may approximate $v_n(q, x)$ as follows:

$$\begin{aligned} v_n(x) &= \frac{q}{(q+r)} 1_{(0,h)} \mathcal{E}_-^+ 1_{(0,h)} \mathcal{E}_+^+ \mathcal{E}_+^- 1_{(0,h)} \mathcal{E}_-^- \mathcal{E}_-^+ 1_{(0,h)} \mathcal{E}_+^+ v_{n-1}(x) \\ &+ O(q^{-1}) \text{ as } q \to +\infty. \end{aligned}$$

Approximate Wiener-Hopf factorization

Notice that $X^{-}_{T_{q+r}}$ admits a semi-explicit factorization. Set

$$\phi_+(\xi) = E[e^{i\xi \bar{X}^{-}_{T_{q+r}}}], \phi_-(\xi) = E[e^{i\xi \underline{X}^{-}_{T_{q+r}}}].$$

$$\phi^+_-(\xi) = \frac{\beta^+_{q+r}}{\beta^+_{q+r} - i\xi}, \phi^-_-(\xi) = \frac{(q+r)(\beta^+_{q+r} - i\xi)}{\beta^+_{q+r}(q+r+\psi_-(\xi))}.$$

Now, we can rewrite the operators $\mathcal{E}^+_-$ and $\mathcal{E}^-_-$ as follows

$$\begin{aligned} \mathcal{E}^+_- u(x) &= E[u(x+\bar{X}^{-}_{T_{q+r}})] = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}\phi^+_-(\xi)\hat{u}(\xi)d\xi \\ \mathcal{E}^-_- u(x) &= E[u(x+\underline{X}^{-}_{T_{q+r}})] = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}\phi^-_-(\xi)\hat{u}(\xi)d\xi. \end{aligned}$$

Approximate Wiener-Hopf factorization

Notice that $X^+_{T_{q+r/2}}$ admits a semi-explicit factorization. Set

$$\phi^+_+(\xi) = E[e^{i\xi \bar{X}^+_{T_{q+r/2}}}], \phi^-_+(\xi) = E[e^{i\xi \underline{X}^+_{T_{q+r/2}}}].$$

$$\phi_+(\xi) = \frac{2(q+r)(-\beta^-_{q+r} + i\xi)}{(2(q+r) + \psi_+(\xi))(-\beta^-_{q+r})}, \phi_-(\xi) = \frac{-\beta^-_{q+r}}{-\beta^-_{q+r} + i\xi}.$$

Now, we can rewrite the operators $\mathcal{E}^+_+$ and $\mathcal{E}^-_+$ as follows

$$\begin{aligned}
\mathcal{E}^+_+ u(x) &= E[u(x + \bar{X}^+_{T_{q+r/2}})] = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} \phi^+_+(\xi) \hat{u}(\xi) d\xi \\
\mathcal{E}^-_+ u(x) &= E[u(x + \underline{X}^+_{T_{q+r/2}})] = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} \phi^-_+(\xi) \hat{u}(\xi) d\xi.
\end{aligned}$$

Outline

Simplified Wiener-Hopf factorization method for barrier options in Lévy models with jumps of bounded variation

Numerical experiments in Kudryavtsev and Luzhetskaya (2020) and Kudryavtsev (2021) for the case of single- and double- barrier options in Lévy models of bounded variation show that the approach similar to the SWHF-method can be a competitor for more accurate constructions of an approximate Wiener-Hopf factorization.

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Numerical examples. Double barrier options

Knock-and-out put option

As a basic example, we consider the knock-and-out put option with the strike K, the lower barrier D, the upper barrier U and time to expiry T.

Model parameters

We take the KoBoL (TSL) model of order $\nu \in (1, 2)$, with the parameters $\sigma = 0$, $\nu = 1.25$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$
($C = 1$, $G = 9$, $M = 8$, $Y = 1.25$ in CGMY parametrization).

Option parameters

the instantaneous interest rate: $r = 0.03$,
time to expiry: $T = 0.1$ year,
the strike price: $K = 3500$,
the barriers: $D = 2800$ and $U = 4200$.

Numerical examples

We check the performance of the SWHF-method against prices obtained by the iterative Wiener-Hopf factorization method developed in Boyarchenko M and Levendorskii S (2011) (BL method) and the methodology for approximating Lévy processes by HEJD processes designed in Crosby J et al (2010) (CC method).

The computations of the option prices were performed in 5 points $x_k = \ln(S/K)(= 0.81, 0.90, 1.00, 1.10, 1.19)$, where S – initial spot price.

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Performance of the SWHF method

Prices: knock-and-out put in KoBoL (CGMY) model

S/K	CC[a] $Nt = 14$	BL[b] $M = 2^{12}$? $N = 80$?	SWHF[c] $M = 2^9$ $N = 5000$	SWHF $M = 2^{10}$ $N = 5000$	SWHF $M = 2^{10}$ $N = 10^4$
81%	10.7739	10.2509	9.9047	10.2306	10.3746
90%	56.5917	56.0175	54.9748	55.3199	55.6443
100%	67.4415	66.5274	66.1113	65.2630	65.5698
110%	45.5662	44.4557	44.5456	44.4710	44.6760
119%	9.1482	8.3171	9.1336	8.5147	8.6425
CPU time	n/a	n/a	1.466 sec	2.908 sec	5.744 sec

[a] The CC-method: Nt – the number of terms in the HEJD model.
[b] The BL-method: M – the number of space points, N – the number of time steps.
[c] The SWHF-method: M – the number of space points, N – the number of time steps.

Performance of the SWHF method

Relative errors w.r.t. BL-method: knock-and-out put in KoBoL (CGMY) model

S/K	CC $Nt = 14$	SWHF $M = 2^9$ $N = 5000$	SWHF $M = 2^{10}$ $N = 5000$	SWHF $M = 2^{10}$ $N = 10^4$
81%	5,1%	-3,4%	-0,2%	1,2%
90%	1,0%	-1,9%	-1,2%	-0,7%
100%	1,4%	-0,6%	-1,9%	-1,4%
110%	2,5%	0,2%	0,0%	0,5%
119%	10,0%	9,8%	2,4%	3,9%
CPU time	n/a	1.466 sec	2.908 sec	5.744 sec

[a] The CC-method: Nt – the number of terms in the HEJD model.

[b] The SWHF-method: M – the number of space points, N – the number of time steps.

Conclusion

- We suggest a new approach for pricing path-dependent options with a payoff depending on the infimum and supremum of a Lévy process at expiry
- The Wiener-Hopf operators can be numerically implemented by using FFT.
- The SWHF-method is rather simpler to implement into program in comparison with existing numerical methods.
- The method suggested makes it easy to implement such a sophisticated tool as the matrix Wiener-Hopf factorization for general Lévy models with jumps of unbounded variation.

Bibliography

Kudryavtsev O. “A simplified Wiener-Hopf factorization method for pricing double barrier options under Lévy processes”, *Computational Management Science* , (2024) 21:37.