Advanced Monte Carlo method for pricing seasoned lookback options under Levy models

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Outline

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Historical background

Option valuation under Lévy processes has been dealt with by a host of researchers.
However, the pricing exotic options in exponential Lévy models still remains a mathematical and computational challenge.

Methods for pricing exotic options: drawbacks

- Monte Carlo methods: *slow*
- Finite difference schemes: *application entails a detailed analysis of the underlying Lévy process*
- Wiener-Hopf methods: *the most efficient in the case of processes with rational characteristic exponents*

The main goal

To suggest a new Monte Carlo method for pricing lookback options.

Outline

Lévy processes: a short reminder

General definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$.

The characteristic exponent of Lévy process

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y\mathbf{1}_{|y|\leq 1})F(dy),$$

where σ^2 is the variance of the Gaussian component, and the Lévy measure $F(dy)$ satisfies $\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}F(dy) < +\infty$.

Examples of Lévy processes

Tempered stable Lévy processes (TSL)

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}],$$

where $\nu_+, \nu_- \in (0,2), \nu_+, \nu_- \neq 1$, $c_+, c_- > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. If $c_- = c_+ = c$ and $\nu_- = \nu_+ = \nu$, then we obtain a KoBoL (CGMY) model.
In the CGMY parametrization $C = c$, $Y = \nu$, $G = \lambda_+$, $M = -\lambda_-$.

Kou model

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where $c_+, c_- \geqslant 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Wiener-Hopf factorization

Let $q > 0$, X_t be a Lévy process with characteristic exponent $\psi(\xi)$, $T_q \sim \mathrm{Exp}\, q$, $\bar{X}_t = \sup_{0 \leqslant s \leqslant t} X_s$ and $\underline{X}_t = \inf_{0 \leqslant s \leqslant t} X_s$ – supremum and infimum processes.

$$\phi_q^+(\xi) = E[e^{i\xi \bar{X}_{T_q}}], \quad \phi_q^-(\xi) = E[e^{i\xi \underline{X}_{T_q}}], \quad \frac{q}{q + \psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi).$$

Introduce the following operators:

$$\mathcal{E}_q^+ g(x) = E[g(x + \bar{X}_{T_q})], \, \mathcal{E}_q^- g(x) = E[g(x + \underline{X}_{T_q})].$$

$$\mathcal{E}_q^\pm g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \phi_q^\pm(\xi) \hat{g}(\xi) d\xi.$$

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“Approximate Wiener-Hopf Monte Carlo”

Kudryavtsev (2019) suggests a generalized approach to the construction of a Monte Carlo method involving approximate factorization for a wide class of Lévy processes.
The main results:

- the MC methods based on time randomization have slow convergence (of order n^{-1});
- a new approach that involves direct simulation of terminal values of the infimum (supremum) process.

Bibliography

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Laplace transform of cdf functions

WH method

- Denote by $F_+(x,T) = \mathbf{P}(\bar{X}_T < x)$ and $F_-(x,T) = \mathbf{P}(\underline{X}_T < x)$
- Apply Laplace transform to $F_+(-x,T)$, $x < 0$:

$$\begin{aligned}\hat{F}_+(-x,q) &= \int_0^{+\infty} e^{-qt} E^x\left[\mathbf{1}_{(-\infty,0)}(\bar{X}_t)\right] dt \\ &= q^{-1} E\left[\mathbf{1}_{(-\infty,0)}(x + \bar{X}_{T_q})\right] = q^{-1}\mathcal{E}_q^+ \mathbf{1}_{(-\infty,0)}(x)\end{aligned}$$

- Apply Laplace transform to $F_-(-x,T)$, $x > 0$:

$$\begin{aligned}\hat{F}_-(-x,q) &= \int_0^{+\infty} e^{-qt} E^x\left[\mathbf{1}_{(-\infty,0)}(\underline{X}_t)\right] dt \\ &= q^{-1} E\left[\mathbf{1}_{(-\infty,0)}(x + \underline{X}_{T_q})\right] = q^{-1}\mathcal{E}_q^- \mathbf{1}_{(-\infty,0)}(x)\end{aligned}$$

Further steps

Key ideas

- Approximate Wiener-Hopf factors $\phi_q^{\pm}(\xi)$ by using the FFT for real-valued functions.
- Apply the Laplace transform to $F_{\pm}(-x, T)$
- Find at q specified by the Gaver-Stehfest algorithm:

$$\hat{F}_+(-x, q) = q^{-1}\mathcal{E}_q^+ \mathbf{1}_{(-\infty,0)}(x) \quad \hat{F}_-(-x, q) = q^{-1}\mathcal{E}_q^- \mathbf{1}_{(-\infty,0)}(x).$$

- Cdf $F_{\pm}(x, T)$ can be recovered from $\hat{F}_{\pm}(-x, q)$ by the Gaver-Stehfest algorithm.
- If the cdf F_X is known then one may simulate X by using samples from $F_X^{-1}(U)$, where U is a uniform distribution on $(0, 1)$.

Numerical Laplace transform inversion: the Gaver-Stehfest algorithm

An approximate formula for $f(\tau)$ can be written as follows

$$f(\tau) \approx \frac{1}{\tau} \sum_{k=1}^{N} \omega_k \cdot \tilde{f}\left(\frac{\alpha_k}{\tau}\right), \qquad 0 < \tau < \infty,$$

$$\begin{aligned} N &= 2n; \\ \alpha_k &= k \ln(2) \\ \omega_k &:= \frac{(-1)^{n+k} \ln(2)}{n!} \sum_{j=[(k+1)/2)]}^{\min\{k,n\}} j^{n+1} C_n^j C_{2j}^j C_j^{k-j}, \end{aligned}$$

where $[x]$ – integer part x и $C_L^K = \frac{L!}{(L-K)!K!}$ – binomial coefficients.

Approximate Wiener-Hopf factorization

The Fast Wiener-Hopf factorization method (FWHF-method)

- In Kudryavtsev and Levendorskiĭ (2009) the fast, accurate and universal numerical method for pricing barrier option under Lévy models was developed.
- In Kudryavtsev (2016) the approximate factorization was generalized; convergence of the method was accelerated.

Reference

KUDRYAVTSEV, O.E., AND S.Z. LEVENDORSKIĬ, "Fast and accurate pricing of barrier options under Levy processes", *J. Finance and Stochastics*, 2009, V. 13, N. 4, 531-562

KUDRYAVTSEV O. Advantages of the Laplace transform approach in pricing first touch digital options in Lévy-driven models. Boletín de la Sociedad Matemática Mexicana, 2016, vol. 22(2), pp. 711–731.

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Lookback options: floating strike

European floating strike lookback put

$$V(T,x) = E^x\left[e^{-rT}(Se^{\bar{X}_T} - Se^{X_T})\right],$$

Seasoned European floating strike lookback put

$$V(T_1, T_2; x, h) = E_{T_1}\left[e^{-r(T_2-T_1)}(Se^{\bar{X}_{T_2}} - Se^{X_{T_2}}) \middle| X_{T_1} = x, \bar{X}_{T_1} = h\right].$$

Set $T = T_2 - T_1$.

$$\begin{aligned} V(T,x) &= E^x\left[e^{-rT}S(e^{\max\{\bar{X}_T, h\}} - e^{X_T})\right] \\ &= E^x\left[e^{-rT}S(e^{\bar{X}_T} - e^{X_T})\right] + \\ &= E^x\left[e^{-rT}(H - Se^{\bar{X}_T})\mathbf{1}_{\{\bar{X}_T < h\}}\right]. \end{aligned}$$

$H(= Se^h)$ – predefined maximum, $E^x\left[e^{-rT}e^{X_T}\right] = e^x$.

Lookback options: fixed strike

European fixed strike lookback put

$$V(T,x) = E^x\left[e^{-rT}(K - Ke^{\underline{X}_T})_+\right],$$

Seasoned European fixed strike lookback put

$$V(T_1, T_2; x, h) = E_{T_1}\left[e^{-r(T_2-T_1)}(K - Ke^{\underline{X}_{T_2}})_+ | X_{T_1} = x, \underline{X}_{T_1} = h\right].$$

Set $T = T_2 - T_1$.

$$\begin{aligned} V(T,x) &= E^x\left[e^{-rT}(K - Ke^{\min\{\underline{X}_T, h\}})_+\right] \\ &= E^x\left[e^{-rT}(K - Ke^{\underline{X}_T})_+\right] + \\ &= E^x\left[e^{-rT}\left((K-H)_+ - (K - Ke^{\underline{X}_T})_+\right)\mathbf{1}_{\{\underline{X}_T > h\}}\right]. \end{aligned}$$

$H(= Ke^h)$ – predefined minimum

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Numerical examples. European Lookbacks

gAWHF&MC-method

The algorithm of the generalized approximate Wiener-Hopf factorization Monte Carlo method was published in Kudryavtsev O.E.(2019). We will refer to it as gAWHF&MC-method.

Bibliography

KUDRYAVTSEV, O.E., "Approximate Wiener–Hopf factorization and Monte Carlo methods for Lévy processes", *Theory Probab. Appl.*, 2019, Vol. 64, No. 2.

Numerical examples

We check the performance of the gAWHF&MC-method against prices obtained by deterministic methods: the FWHF&GS-method from Kudryavtsev O., Levendorskii S. (2011) and the ParaiLT-method from Boyarchenko S. I., Levendorskii S. Z.(2013).

Bibliography

KUDRYAVTSEV, O.E., AND S.Z. LEVENDORSKIĬ, "Efficient pricing options with barrier and lookback features under Levy processes", Working paper, 2011, 29 pp. Available at SSRN.

BOYARCHENKO S. I., LEVENDORSKII S. Z. "Efficient Laplace inversion, Wiener-Hopf factorization and pricing lookbacks", *International Journal of Theoretical and Applied Finance*, 2013, vol. 16(3), 1350011.

Experiment setup

We consider European fixed strike lookback put options under the TSL model, and use the same parameters of the KoBoL (CGMY) process as in Boyarchenko S. I., Levendorskii S. Z.(2013)://
$c = 0.2395$, $\lambda_+ = 3.0$, $\lambda_- = -10.0$, $\nu = 1.2$ ($C = 0.2395$, $G = 3.0$, $M = 10.0$, $Y = 1.2$ in CGMY parametrization).

The remaining parameters are strike price $K = 100$, the dividend rate $d = 0$ and interest rate $r = 0.04$. The drift parameter μ is fixed by EMM-requirement. We consider 2 maturities $T = 0.1$ (short) and $T = 2$ (long).

The computations performed in 10 points
$x_k = \ln(S/K)(= 0.02; 0.04; ...; 0.2)$, where S – initial spot price.

PC characteristics: Intel Core(TM)i5 CPU, 1.7GHz, 4 GB RAM, Windows 7 Professional with 64-bit

Numerical examples

European lookbacks

For verification of the accuracy of our method, we calculate prices for the fixed strike lookback put by using the gAWHF&MC-method, the FWHF&GS-method from the ParaiLT-method.

The prices of the FWHF&GS-method were obtained using the code implemented into the program platform Premia (www.premia.fr).

The prices of the ParaiLT-method were taken from the table 3 Boyarchenko S. I., Levendorskii S. Z.(2013), as well as the benchmarks.

gAWHF&MC-prices converge very fast and agree with the benchmarks. All the methods are in agreement. gAWHF&MC-method for pricing lookback options could be considered as a competitor to the deterministic methods.

Convergence of gAWHF&MC

Fixed strike lookback put prices and MC-errors. Short maturity

Parameters	$h = 0.001,\ N = 10^4$		$h = 0.001,\ N = 10^5$		$h = 0.001,\ N = 10^6$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	5.34748	0.147	5.38840	0.046	5.39141	0.015
0.04	4.21819	0.140	4.25655	0.044	4.26282	0.014
0.06	3.34641	0.132	3.37984	0.042	3.38694	0.013
0.08	2.67288	0.123	2.69821	0.039	2.70609	0.012
...	...	...	...	...	...	...
0.2	0.81726	0.079	0.81431	0.025	0.81986	0.008
Parameters	$h = 0.0005,\ N = 10^4$		$h = 0.0005,\ N = 10^5$		$h = 0.0005,\ N = 10^6$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	5.33876	0.147	5.37970	0.046	5.38274	0.015
0.04	4.21152	0.140	4.24982	0.044	4.25610	0.014
0.06	3.34122	0.132	3.37463	0.042	3.38172	0.013
0.08	2.66890	0.123	2.69414	0.039	2.70203	0.012
...	...	...	...	...	...	...
0.2	0.81631	0.079	0.81333	0.025	0.81888	0.008
Parameters	$h = 0.0001,\ N = 10^4$		$h = 0.0001,\ N = 10^5$		$h = 0.0001,\ N = 10^6$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	5.33171	0.147	5.37264	0.046	5.37571	0.015
0.04	4.20611	0.140	4.24435	0.044	4.25064	0.014
0.06	3.33699	0.132	3.37037	0.042	3.37745	0.013
0.08	2.66565	0.123	2.69079	0.039	2.69869	0.012
...	...	...	...	...	...	...
0.2	0.81547	0.079	0.81246	0.025	0.81801	0.008

Convergence of gAWHF&MC

Fixed strike lookback put prices and MC-errors. Long maturity

Parameters	$h = 0.001,\ N = 10^5$		$h = 0.001,\ N = 10^6$		$h = 0.001,\ N = 10^7$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	28.29943	0.125	28.27669	0.040	28.26846	0.013
0.04	27.15904	0.126	27.13597	0.040	27.12789	0.013
0.06	26.05742	0.127	26.03453	0.040	26.02668	0.013
0.08	24.99020	0.128	24.96794	0.040	24.96014	0.013
...	...	...	...	...	...	...
0.2	19.22577	0.125	19.21361	0.040	19.20667	0.013
Parameters	$h = 0.0005,\ N = 10^5$		$h = 0.0005,\ N = 10^6$		$h = 0.0005,\ N = 10^7$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	28.29156	0.125	28.26881	0.040	28.26058	0.013
0.04	27.15145	0.126	27.12838	0.040	27.12031	0.013
0.06	26.05008	0.127	26.02719	0.040	26.01934	0.013
0.08	24.98308	0.128	24.96082	0.040	24.95302	0.013
...	...	...	...	...	...	...
0.2	19.21983	0.125	19.20769	0.040	19.20075	0.013
Parameters	$h = 0.0001,\ N = 10^5$		$h = 0.0001,\ N = 10^6$		$h = 0.0001,\ N = 10^7$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	28.28453	0.125	28.26178	0.040	28.25354	0.013
0.04	27.14467	0.126	27.12160	0.040	27.11352	0.013
0.06	26.04351	0.127	26.02061	0.040	26.01277	0.013
0.08	24.97671	0.128	24.95444	0.040	24.94665	0.013
...	...	...	...	...	...	...
0.2	19.21448	0.125	19.20235	0.040	19.19541	0.013

Comparison of gAWHF&MC with deterministic methods

Errors. Short maturity

x	0.02	0.04	0.06	0.08	...	0.20	Time
V_{put}	5.37205	4.24803	3.37586	2.69765	...	0.81512	400-1900
Para iLT							
ϵ=E-01	-0.053	-0.038	-0.028	-0.014	...	0.0064	0.03-0.15
ϵ=E-02	-0.0054	-0.0041	-0.0032	-0.0025	...	-0.0007	0.55-1.49
FWHF&GS_7							
$h = 0.001$	0.00669	0.00633	0.00546	0.00447	...	0.00094	0.078
$h = 0.0005$	0.00352	0.00326	0.00276	0.00223	...	0.00043	0.188
$h = 0.0002$	0.00124	0.00111	0.00091	0.00072	...	0.00009	0.39
gAWHF&MC							
$h = 0.001$							
$N = 10^4$	-0.0246	-0.02984	-0.02945	-0.02478	...	0.00215	0.156
$N = 10^5$	0.0164	0.00852	0.00398	0.00055	...	-0.00081	0.1880
$N = 10^6$	0.0194	0.01479	0.01108	0.00843	...	0.00474	0.797
$h = 0.0005$							
$N = 10^4$	-0.0333	-0.03650	-0.03464	-0.02875	...	0.00120	0.266
$N = 10^5$	0.0077	0.00180	-0.00123	-0.00352	...	-0.00178	0.328
$N = 10^6$	0.0107	0.00808	0.00586	0.00438	...	0.00377	0.906
$h = 0.0001$							
$N = 10^4$	-0.0403	-0.04191	-0.03887	-0.03201	...	0.00036	1.125
$N = 10^5$	0.0006	-0.00367	-0.00548	-0.00686	...	-0.00266	1.172
$N = 10^6$	0.0037	0.00262	0.00160	0.00104	...	0.00289	1.813

Comparison of gAWHF&MC with deterministic methods

Errors. Long maturity

x	0.02	0.04	0.06	0.08	...	0.20	Time
V_{put}	28.25454	27.11439	26.01360	24.94750	...	19.19671	873-2762
Para iLT							
ϵ=E-03	-0.068	-0.062	-0.057	-0.052	...	-0.029	0.23-1.13
ϵ=E-04	-0.0043	-0.0058	-0.0060	-0.0057	...	-0.0033	1.36-5.1
FWHF& GS_7							
$h = 0.001$	0.02686	0.02731	0.02770	0.02790	...	0.0281	0.093
$h = 0.0005$	0.01266	0.01291	0.01300	0.01310	...	0.0132	0.187
$h = 0.0002$	0.00196	0.00221	0.00230	0.00250	...	0.0030	0.359
gAWHF&MC							
$h = 0.001$							
$N = 10^5$	0.0449	0.0446	0.0438	0.0427	...	0.0291	0.218
$N = 10^6$	0.0222	0.0216	0.0209	0.0204	...	0.0169	1.062
$N = 10^7$	0.0139	0.0135	0.0131	0.0126	...	0.0100	7.59
$h = 0.0005$							
$N = 10^5$	0.0370	0.0371	0.0365	0.0356	...	0.0231	0.359
$N = 10^6$	0.0143	0.0140	0.0136	0.0133	...	0.0110	1.062
$N = 10^7$	0.0060	0.0059	0.0057	0.0055	...	0.0040	7.98
$h = 0.0001$							
$N = 10^5$	0.0300	0.0303	0.0299	0.0292	...	0.0178	1.156
$N = 10^6$	0.0072	0.0072	0.0070	0.0069	...	0.0056	1.859
$N = 10^7$	-0.0010	-0.0009	-0.0008	-0.0009	...	-0.0013	9.53