Risk measures for variable annuities under Lévy processes*

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Outline

VaR: Main definitions

VaR

The Value-at-Risk measures the potential loss in value of a risky asset or portfolio at the end of a specified trading horizon with a given confidence level.

VaR and European digital options

European digital contract pays \$1 at the maturity date T if the underlying asset price S_T crossed a prefixed barrier H, and nothing otherwise.

References

Basel Committee: Overview of the amendment to the capital accord to incorporate market risks. *Basel Committee on Banking Supervision*, 1996.

Kim, Y.S., Rachev, S., Bianchi, M.S., Fabozzi, F.J. Computing VaR and AVar in Infinitely Divisible Distributions. *Probability and Mathematical Statistics*, 2010

Computing VaR and CTE in stochastic models

Let the random variable X represents the loss of a portfolio. Let $F_X(x) = \mathsf{P}(X < x)$, $p_X(x) = \frac{d}{dx}F_X(x)$, $\phi(\xi) = E[e^{i\xi X}]$ are cdf, pdf and chf of X, respectively.
The VaR and CTE of X at tail probability α is defined as follows.

$$\mathrm{VaR}_\alpha(X) = \inf\{y \in \mathsf{R} | F_X(y) \geqslant \alpha\},$$

$$CTE_\alpha(X) := \mathbb{E}\left[X | X > \mathrm{VaR}_\alpha(X)\right].$$

If $F_X(x)$ is continuous, then

$$F_X(x) = E[1_{X<x}] = \int_{-\infty}^{x} p_X(y)dy.$$

and the following formula is valid:

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$

Variable annuity

Definition

A variable annuity is a long-term investment for retirement which benefits offered by insurance companies are usually protected via different mechanisms such as e.g. guaranteed minimum maturity benefits (GMMBs). The computation of the corresponding risk measures is an important issue for the practitioners in risk management.

GMMB

The GMMB riders provide minimum guarantees to protect the investment account of the policyholder. Denoting by τ_x the future lifetime of a policyholder at the age x, the future payment made by the insurer is

$$(G - F_T)^+ 1_{\{\tau_x > T\}}$$

at maturity T for GMMBs, where G is the guarantee level expressed as a percentage of the initial fund value F_0.

Variable annuity

The net liability of GMMBs is
$L_0 := e^{-rT}(G - F_T)^+ 1_{\{\tau_x > T\}} - \int_0^{T \wedge \tau_x} e^{-rs} M_s^e \mathrm{d}s,$
where $F_t := e^{-mt} S_t$ is the current fund value;
$M_t^e := m_e F_t$ is the margin offset income;
S_t is the underlying equity value.

Methods for computing risk measures

The quantile risk measures of the net liabilities of GMMBs have been evaluated by analytical methods, the method based on identity in law, and using conditional moment matching for the Geometric Brownian motion.

The main goal

We propose an efficient numerical method based on the frame projection approach (the PROJ-method) for calculating the VaR and CTE for variable annuities for a wide class of Lévy processes.

Outline

Lévy processes: a short reminder

General definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X_t}] = e^{-t\psi(\xi)}$.

The characteristic exponent of Lévy process

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y 1_{|y|\le 1})\Pi(dy),$$

where σ^2 is the variance of the Gaussian component, and the Lévy measure $\Pi(dy)$ satisfies $\int_{\mathbb{R}\setminus\{0\}} \min\{1, y^2\}\Pi(dy) < +\infty$.
If $\Pi(dx) = \pi(x)dx$, $\pi(x)$ – Lévy density.

Examples of Lévy processes, $\Pi(\mathbb{R}) < \infty$

Jump diffusion

$X_t = \gamma_0 t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$, where W_t – Brownian motion, N_t – Poisson process with intensity λ, and Y_i – i.i.d of jumps.

Kou model

The Lévy density $\pi(x)$, is of the form

$$\pi(x) = (1-p)\lambda\Lambda_- e^{\Lambda_- x} 1_{\{x<0\}} + p\lambda\Lambda_+ e^{-\Lambda_+ x} 1_{\{x>0\}}.$$

where $\Lambda_- > 0, \Lambda_+ > 1, 0 < p < 1, \lambda > 0$.
If we set $c_+ = (1-p)\lambda\Lambda_-$, $c_- = p\lambda\Lambda_+$, $\lambda_+ = \Lambda_-$, $\lambda_- = -\Lambda_+$, then

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where $\sigma > 0$, $\mu = \gamma_0 - \int_{-1}^{1} x\Pi(dx)$, $c_\pm > 0$ and $\lambda_- < -1 < 0 < \lambda_+$.

Examples of Lévy processes, $\Pi(\mathbb{R}) = \infty$

Tempered stable Lévy processes (TSL)

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}],$$

where $\nu_+, \nu_- \in (0, 2), \nu_+, \nu_- \neq 1$, $c_+, c_- > 0$, $\mu \in \mathbb{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

$$\pi(x) = c_+ e^{\lambda_+ x}|x|^{-\nu_+ - 1}1_{\{x<0\}} + c_- e^{\lambda_- x}|x|^{-\nu_- - 1}1_{\{x>0\}}.$$

If $c_- = c_+ = c$ and $\nu_- = \nu_+ = \nu$, then we obtain a KoBoL (CGMY) model.
In the CGMY parametrization $C = c$, $Y = \nu$, $G = \lambda_+$, $M = -\lambda_-$.

Outline

The key ideas for the VaR computation

The key quantity for computing $VaR_\alpha(L_0)$

$$\mathrm{P}_0\left(T,G,w\right)=P\left\{L_0:=e^{-rT}(G-F_T)^+-\int_0^T e^{-rs}m_eF_sds>w\right\}.$$

To compute $VaR_\alpha(L_0)$ we find a w^* such that ${}_Tp_xP_0\left(T,G,w^*\right)=1-\alpha$, where ${}_Tp_x$ is the probability that a policyholder at age x will survive T units of time, $x,T>0$.
$F_t=F_0e^{X_t-mt}$, X_t - a Lévy process

The problem transformation

$$P_0\left(T,G,w^*\right)=\mathrm{P}\left\{e^{-rT}F_T+m_e\int_0^T e^{-rs}F_sds<\left(e^{-rT}G-w^*\right)\right\}.$$

Now, we need to find the cdf of $L_0'=e^{-rT}F_T+m_e\int_0^T e^{-rs}F_sds$.

Approximation of L_0'

We set $F_t' = e^{-rt} F_t$ and rewrite L_0' as $L_0' = F_T' + m_e \int_0^T F_s' ds$.
Let F_t' be a Lévy process with the characteristic exponent ψ.

Time discretization

Select the following partitions of $[0, T]$:
$\mathrm{T} = \{t_0, t_1, \ldots, t_M\}$, where $t_j = j\Delta t = j\frac{T}{M}$,
$\mathrm{T}^\star = \{t_0^\star, t_1^\star, \ldots, t_{M-1}^\star\}$, where $t_j < t_j^\star < t_{j+1}, j = 0, \ldots, M-1$.
We choose the points $t_j^\star$ in such way that

$$E\left[\int_{t_{j-1}}^{t_j} F_s' ds - \int_{t_{j-1}}^{t_j} F_{t_j^\star}' ds\right] = 0.$$

Since $E[F_s'] = \exp(-s\psi(-i))]$, we obtain the equation for $t_j^\star$:
$\int_{t_{j-1}}^{t_j} \exp(-s\psi(-i)]ds - \int_{t_{j-1}}^{t_j} \exp(-t_j^\star\psi(-i))ds = 0$.

Approximation of L_0'

If $\psi(-i) = 0$, then $t_j^\star$ can be set to any value of the interval $[t_{j-1}, t_j]$. Otherwise, we have

$$t_j^\star = t_{j-1} + \frac{1}{-\psi(-i)} \log\left(\frac{1 - \exp(-\Delta t \psi(-i))}{\psi(-i)\Delta t}\right),$$

Estimating the integral $\int_0^T F_s' ds$ with the Riemann sum, we may write the approximation L_M^ω for L_0' as $L_M^\omega = \sum_{j=0}^{M-1} \omega_j F'_{t_j^\star} + \omega_M F'_{t_M}$, where $\omega_M = 1$ and the weights $\omega_j = m_e \cdot \Delta t$, $j = 0, \ldots, M-1$.

If the density of L_M^ω is known, say $p_{L_M^\omega}$, then

$$P_0\left(T, G, VaR_\alpha\left(L_0\right)\right) \approx \int_0^W p_{L_M^\omega}(u) du,$$

where $W = e^{-rT} G - VaR_\alpha\left(L_0\right)$.

Iterative scheme

Proposition

Fix a set of positive weights $\omega = \{\omega_j\}_{j=0}^M$ *in the formula for* L'_M *with* $\omega_M = 1$. *Introduce* $R_M = \log(F'_{t_M}/F'_{t^\star_{M-1}})$, $R_0 = \log(F'_{t^\star_0}/F_0)$, *and* $R_j = \log(F'_{t^\star_j}/F'_{t^\star_{j-1}})$, $j = 1, \ldots, M-1$. *Set* $Y_M := R_M$, *and define recursively* $Y_j = R_j + Z_{j+1}, j = 1, \ldots, M-1$, $Y_0 = R_0 + Z_1$, *where* $Z_j := \log(\omega_{j-1} + \exp(Y_j))$. *Then*

$$L'_M \equiv F_0 \exp(Y_0),$$

where the ChF of Y_0 *can be found iteratively as follows:*

$$\begin{aligned}
\phi_{Y_M}(\xi) &= \phi_{R_M}(\xi); \phi_{Z_j}(\xi) = \phi_{\log(\omega_{j-1}+\exp(Y_j))}(\xi), \quad j = M, ..., 1; \\
\phi_{Y_j}(\xi) &= \phi_{R_j}(\xi)\phi_{Z_{j+1}}(\xi), \quad j = M-1, ..., 0.
\end{aligned}$$

Notice that $\phi_{Z_j}(\xi) = \int_{\mathbb{R}} (e^y + \omega_{j-1})^{i\xi} p_{Y_j}(y) dy$.

Computing risk measures

Computing VaR

According to the Proposition,

$$P_0\left(T, G, VaR_\alpha\left(L_0\right)\right) \approx \int_{-\infty}^{y^*} p_{Y_0}(y)dy,$$

where $y^* = \log(W/F_0)$ should be found from the equation

$$\int_{-\infty}^{y^*} p_{Y_0}(y)dy = \frac{1-\alpha}{{}_T px}.$$

Computing CTE

Further, we have

$$CTE_\alpha\left(L_0\right) \approx Ge^{-rT} - \frac{{}_T px}{1-\alpha} F_0 \int_{-\infty}^{y^*} e^y p_{Y_0}(y)dy.$$

Outline

The frame projection approach (the PROJ-method)

If there is no an explicit formula for the pdf p_{X_T}, it can be recovered by inverting the chf $\phi_{X_T}(\xi)$ using the inverse Fourier transform:
$p_{X_T}(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi}\phi_{X_T}(\xi)d\xi.$

In series of papers the frame projection approach (PROJ) was developed. In particular, in Kirkby (2016) the approach was applied for robust pricing Asian under exponential Lévy models.

Coefficient functionals of the orthogonally projected transition density are given by its convolution with a dual B-spline scaling function of the second order, using the characteristic function of the underlying asset.

Reference

Kirkby, J. L. An efficient transform method for Asian option pricing. *SIAM Journal on Financial Mathematics*, 2016.

The frame projection

The B-spline bases of order p are of particular interest and can be derived as follows. Starting with the Haar scaling function defined by $\varphi^{[0]}(y) := 1_{[-\frac{1}{2},\frac{1}{2}]}(y)$, the p-th order B-spline scaling functions are derived successively by the convolution

$$\varphi^{[p]}(x) = \varphi^{[0]} \star \varphi^{[p-1]}(x) = \int_{-\infty}^{\infty} \varphi^{[p-1]}(y-x) 1_{[-\frac{1}{2},\frac{1}{2}]}(y) dy.$$

Denote by $\phi(\nu)$ a symmetric generator of the B-spline basis. For a fixed sampling step $h > 0$, we consider a space of compactly supported basis elements

$$\phi_{h,k}(\nu) := \phi((\nu - x_k)/h),$$

where $x_n = x_0 + nh, \quad n \in \mathbb{Z}$.
Let $\widetilde{\phi}_{h,k}$ be the dual basis with a generator $\widetilde{\phi}$.

The frame projection

Riesz basis

Define $V_{\varphi_h} = \{f(x) = \sum_{n\in Z} c_{n,h}\phi_{h,n}(x) | \{c_{n,h}\}_{n\in Z} \in l_2(Z)\}$. The system of functions $\{\phi_{h,n}\}$ forms the Riesz basis of V_{φ_h} since it satisfies the requirement

$$A\|c\|^2_{l^2(Z)} \leq \sum_{n\in Z} \|c_n\phi_{h,n}(x)\|^2_{L^2(R)} \leq B\|c\|^2_{l^2(Z)}, \quad \forall c \in l^2(Z), (*)$$

for some constants A and B such that $0 < A \leq B$.

Frames

Recall that the collection of functions $\{\phi_{h,n}\}_{n\in Z}$ constitutes a frame of the function space V_{φ_h}, if condition $(*)$ is relaxed as follows

$$A\|f\|^2_{L^2(R)} \leq \sum_{n\in Z} |\langle f, \varphi_{h,n}\rangle_{L^2(R)}|^2 \leq B\|f\|^2_{L^2(R)}, \quad \forall f \in V_{\varphi_h}.$$

The frame projection

For a fixed $h > 0$, and the generator $\phi(\nu)$, we obtain

$$f(\nu) \approx \sum_{k=1}^{N} \left(\frac{1}{h} \int_{-\infty}^{+\infty} f(y) \widetilde{\phi}_{h,k}(y) dy \right) \phi_{h,k}(\nu)$$

which provides the L_2 projection restricted to $\{\phi_{h,k}(\nu)\}_{k=1}^{k=N}$.

$\phi = \phi^{[3]}$

With $p = 3$, the cubic B-spline writes down as

$$\varphi^{[3]}(y) = \begin{cases} (2+y)^3/6, & y \in [-2,-1] \\ 2/3 - y^2 - y^3/2, & y \in [-1,0] \\ 2/3 - y^2 + y^3/2, & y \in [0,1] \\ (2-y)^3/6, & y \in [1,2]\,. \end{cases}$$

Recovery of $p_{Y_j}(x)$

$$p_{Y_j}(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi} \phi_{Y_j}(\xi) d\xi \approx \sum_{n=1}^{N} c^j_{n,h} \phi_{h,n}(x)$$

Using the Fourier transform technique, we obtain

$$\begin{aligned}
c^j_{n,h} &= \frac{1}{h} \int_{-\infty}^{+\infty} \widetilde{\varphi}\Big(\frac{y - x_n}{h}\Big) \left((2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-iy\xi} \phi_{Y_j}(\xi) d\xi \right) dy. \\
&= (2\pi)^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \widetilde{\varphi}(z) e^{-i(zh+x_n)\xi} \phi_{Y_j}(\xi) dz d\xi \\
&= \pi^{-1} \,\mathrm{Re} \int_0^{+\infty} e^{-ix_n\xi} \widehat{\widetilde{\varphi}}(h\xi) \phi_{Y_j}(\xi) d\xi, \\
\widehat{\widetilde{\phi}}(\xi) &= \widehat{\widetilde{\phi}}^{[3]}(\xi) = \frac{2520 \sin^4(\xi/2)/(\xi/2)^4}{(1208 + 1191 \cos(\xi) + 120 \cos(2\xi) + \cos(3\xi))}.
\end{aligned}$$

Recovery of $\phi_{Y_j}(\xi)$

Recovery of $\phi_{Z_j}(\xi)$

$$
\begin{aligned}
\phi_{Z_j}(\xi) &= \int_{\mathbb{R}} (\omega_{j-1} + e^y)^{i\xi} p_{Y_j}(y) dy \\
&\approx \int_{\mathbb{R}} (\omega_{j-1} + e^y)^{i\xi} \left(\sum_{n=1}^{N} c^j_{n,h} \phi_{h,n}(y) \right) dy \\
&= \left(z = \log(e^y + \omega_{j-1}) \right) \\
&= \int_0^{+\infty} e^{iz\xi} \left(\sum_{n=1}^{N} c^j_{n,h} \phi_{h,n}(\log(e^z - \omega_{j-1})) \right) \frac{e^z}{e^z - \omega_0} dz.
\end{aligned}
$$

The next step:

$$
\bar{\phi}_{Y_{j-1}}(\xi) := \bar{\phi}_{Z_j}(\xi) \phi_{R_{j-1}}(\xi).
$$

Computing VaR

To find numerically $y^\star$ as the solution to the equation

$$\int_{-\infty}^{\widetilde{y}} p_{Y_0}(y)dy = \frac{1-\alpha}{{}_T p x}$$

with respect to $\widetilde{y}$, we will apply Newton's method.

Newton's method for solving $g(y) = \gamma$, where $g \in C^1(\mathrm{R})$

The algorithm of sequential iterations in Newton's method is as follows:

$$y_{n+1} = y_n + \frac{\gamma - g(y_n)}{g'(y_n)}.$$

In our case, we have

$$\widetilde{y}_{n+1} = \widetilde{y}_n + \frac{\gamma - \int_{-\infty}^{\widetilde{y}_n} p_{Y_0}(y)dy}{p_{Y_0}(\widetilde{y}_n)}.$$

Outline

Numerical examples. Geometrical Brownian motion model

As a basic example, we consider the GMMBs with the probability ${}_T p_x = 0.757$ and the risk tolerance levels $\alpha = 0.9$ and $\alpha = 0.95$.

Model parameters

We take the GBM model with the parameters $\sigma = 0.3$, $\mu = 0.09$.

Variable annuity parameters

the instantaneous interest rate: $r = 0.04$,
time to expiry: $T = 10$ years,
the guarantee level: $G_0 = 100$,
the initial fund value $F_0 = 100$,
the annualized mortality rate: $m = 0.01$,
the GMMB coefficient $m_e = 0.003$.

Performance of the PROJ-method

Value-at-Risk and CTE in the Geometrical Brownian motion model

Таблица: $V_\alpha(L_e)$ and $CTE_\alpha(L_e)$ in the GBM model: values obtained by the FV-method and PROJ-method. Convergence in N.

	FV	PROJ	PROJ	PROJ	PROJ
N		2^5	2^6	2^6	2^7
N_0		2^6	2^6	2^7	2^7
M		30	30	30	30
$V_{90\%}$	12.55037	12.84139	12.60686	12.55385	12.55332
$V_{95\%}$	28.93573	29.18902	28.98381	28.93585	28.93772
$CTE_{90\%}$	30.29649	30.72038	30.33158	30.29656	30.29847
$CTE_{95\%}$	40.04152	40.61308	40.05718	40.03978	40.04298
CPU time		0.002 s	0.002 s	0.004 s	0.004 s

We used VaR and CTE obtained in Feng and Volkmer (2012) as the benchmark.

Performance of the PROJ-method

Value-at-Risk and CTE in the Geometrical Brownian motion model

Таблица: $V_\alpha(L_e)$ and $CTE_\alpha(L_e)$ in the GBM model: absolute and relative errors of PROJ-method w.r.t. FV-method. Convergence in N.

	PROJ	PROJ	PROJ	PROJ
N	2^5	2^6	2^6	2^7
N_0	2^6	2^6	2^7	2^7
M	30	30	30	30
$V_{90\%}$	0.2910(2.31%)	0.0565(0.45%)	0.0035(0.0277%)	0.0030(0.024%)
$V_{95\%}$	0.2533(0.9%)	0.0481(0.17%)	0.0001(0.0004%)	0.0020(0.007%)
$CTE_{90\%}$	0.4239(1.4%)	0.0351(0.12%)	0.0001(0.0002%)	0.0020(0.007%)
$CTE_{95\%}$	0.5716(1.4%)	0.0157(0.04%)	-0.0017(-0.004%)	0.0015(0.004%)
CPU time	0.002 s	0.002 s	0.004 s	0.004 s

We used VaR and CTE obtained in Feng and Volkmer (2012) as the benchmark.

Numerical examples. The CGMY model

As a basic example, we consider the GMMBs with the probability ${}_T p_x = 0.757$ and the risk tolerance level $\alpha = 0.95$.

Model parameters

According to Kélani, A. and Quittard-Pinon, F. (2017), we use the parameters of the CGMY model

$$C = 0.6235, G = 21.0775, M = 39.5137, Y = 0.8, \mu = 0.2799$$

Variable annuity parameters

the instantaneous interest rate: $r = 0.04$,
time to expiry: $T = 10$ years,
the guarantee level: $G_0 = 125$,
the initial fund value $F_0 = 100$,
the annualized mortality rate: $m = 0.01$,
the GMMB coefficient $m_e = 0.003$.

Performance of the PROJ-method in the CGMY model

We checked the performance of the PROJ-method against the Monte Carlo method (MC-method).

Value-at-Risk and CTE in the CGMY model

Таблица: $V_\alpha(L_e)$ and $CTE_\alpha(L_e)$ in the CGMY model: values obtained by the MC-method and PROJ-method. Convergence in N and M.

	MC	PROJ	PROJ	PROJ	PROJ
N	$3 \cdot 10^6$	2^5	2^6	2^7	2^8
N_0	—	2^6	2^7	2^7	2^9
M	500	30	30	30	100
$V_{95\%}$	2.4096	2.3702	2.4285	2.4284	2.4286
$CTE_{95\%}$	16.6597	16.7838	16.6736	16.6740	16.6739
CPU time	10hrs	0.002 s	0.003 s	0.004 s	0.04 s

Performance of the PROJ-method in the CGMY model

Value-at-Risk and CTE in the CGMY model: errors

Таблица: $V_\alpha(L_e)$ and $CTE_\alpha(L_e)$ in the CGMY model: absolute and relative errors of PROJ-method w.r.t. MC-method. Convergence in N and M.

	MC	PROJ	PROJ	PROJ	PROJ
N	$3 \cdot 10^6$	2^5	2^6	2^7	2^8
N_0	–	2^6	2^7	2^7	2^9
M	500	30	30	30	100
$V_{95\%}$	0.081(3%)	-0.039(-1.6%)	0.019(0.8%)	0.019(0.8%)	0.019(0.8%)
$CTE_{95\%}$	0.019(0.1%)	0.124(0.8%)	0.014(0.1%)	0.014(0.1%)	0.014(0.1%)
CPU time	10 hrs	0.002 s	0.003 s	0.004 s	0.04 s

GBM vs the CGMY model

Finally, we study the difference between the calculated risk measures in the CGMY model and the Gaussian Lévy model calibrated to the same data set in Kélani, A. and Quittard-Pinon, F. (2017). The GBM parameters: $\sigma = 0.1473$, $\mu = 0.0962$. We introduce the minimum guarantee level G_{min} required for VaR to be positive.

Value-at-Risk and CTE obtained by the PROJ-method: the CGMY model vs. the GBM model.

	CGMY	GBM	Δ_{abs}	Δ_{rel}
N	2^8	2^9		
N_0	2^9	2^{10}		
M	100	100		
$V_{95\%}$	2.4286	1.9175	-0.5111	-21%
$CTE_{95\%}$	16.6739	15.7539	-0.9200	-6%
G_{min}	121.37693	122.13944	0.7625	1%

Conclusion

- In the proposed approach, the probability density of the net liabilities is approximated using the theory of frames and Riesz bases.
- We approximate the integral of the exponential Lévy process by a discrete sum whose expectation coincides with the expected value of the desired integral.
- To find VaR as a quantile of the loss distribution, we numerically solve the equation for the corresponding CDF using Newton's method adapted to the probability density approximation by B-splines.
- Once the VaR is found, we calculate the CTE using integration by parts, again taking advantage of the properties of cubic B-splines.
- Numerical experiments on the application of the developed method for the Black-Scholes and CGMY models clearly demonstrate its high accuracy and speed.