New Monte Carlo method
for pricing lookback options in Lévy models

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Outline

1. The main goal
2. Lévy processes: a short reminder
3. Monte Carlo method and Wiener-Hopf factorization
4. Further steps: numerical Laplace transform inversion
5. A generalization of the AWHMC method
6. Lookback options
7. Numerical examples

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Historical background

Option valuation under Lévy processes has been dealt with by a host of researchers.
However, the pricing options in exponential Lévy models still remains a mathematical and computational challenge.

Methods for pricing exotic options: drawbacks

- Monte Carlo methods: *slow*
- Finite difference schemes: *application entails a detailed analysis of the underlying Lévy process*
- Wiener-Hopf methods: *the most efficient in the case of processes with rational characteristic exponents*

The main goal

To suggest a new Monte Carlo method for a wide class of Lévy processes.

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Lévy processes: a short reminder

General definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$.

The characteristic exponent of Lévy process

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y|\leq 1})F(dy),$$

where σ^2 is the variance of the Gaussian component, and the Lévy measure $F(dy)$ satisfies $\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}F(dy) < +\infty$.

Examples of Lévy processes

Tempered stable Lévy processes (TSL)

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}],$$

where $\nu_+, \nu_- \in (0,2), \nu_+, \nu_- \neq 1$, $c_+, c_- > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. If $c_- = c_+ = c$ and $\nu_- = \nu_+ = \nu$, then we obtain a KoBoL (CGMY) model.
In the CGMY parametrization $C = c$, $Y = \nu$, $G = \lambda_+$, $M = -\lambda_-$.

Kou model

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where $c_+, c_- \geqslant 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Wiener-Hopf factorization

Let $q > 0$, X_t be a Lévy process with characteristic exponent $\psi(\xi)$, $T_q \sim \mathrm{Exp}\, q$, $\bar{X}_t = \sup_{0\leqslant s\leqslant t} X_s$ and $\underline{X}_t = \inf_{0\leqslant s\leqslant t} X_s$ – supremum and infimum processes.

$$\phi_q^+(\xi) = E[e^{i\xi\bar{X}_{T_q}}], \quad \phi_q^-(\xi) = E[e^{i\xi\underline{X}_{T_q}}], \quad \frac{q}{q+\psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi).$$

Introduce the following operators:

$$\mathcal{E}_q g(x) = E^x[\int_0^{+\infty} qe^{-qt} g(X_t)dt] = E^x[g(X_{T_q})].$$

$$\mathcal{E}_q^+ g(x) = E^x[\int_0^{+\infty} qe^{-qt} g(\bar{X}_t)dt] = E^x[g(\bar{X}_{T_q})].$$

$$\mathcal{E}_q^- g(x) = E^x[\int_0^{+\infty} qe^{-qt} g(\underline{X}_t)dt] = E^x[g(\underline{X}_{T_q})].$$

$\mathcal{E}$ and $\mathcal{E}^{\pm}$ as PDO

$$\mathcal{E}_q g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi} q(q+\psi(\xi))^{-1}\hat{g}(\xi)d\xi,$$

$$\mathcal{E}_q^{\pm} g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\phi_q^{\pm}(\xi)\hat{g}(\xi)d\xi.$$

WHF in an operator form: $\mathcal{E} = \mathcal{E}^{+}\mathcal{E}^{-} = \mathcal{E}^{-}\mathcal{E}^{+}$.

$\mathcal{E}_q$ and $\mathcal{E}_q^{\pm}$ as convolution operators

Operators $\mathcal{E}_q$ and $\mathcal{E}_q^{\pm}$ admit the following interpretation:

$$\mathcal{E}_q g(x) = \int_{-\infty}^{+\infty} g(x+y)P(y)dy, \quad \mathcal{E}_q^{\pm} g(x) = \int_{-\infty}^{+\infty} g(x+y)P_{\pm}(y)dy,$$

where $P(y)$, $P_{\pm}(y)$ are probability densities with

$$P_{\pm}(y) = 0, \quad \forall \pm y < 0.$$

Useful facts and relations

Let X_t – Lévy process, and $T_q \sim \mathrm{Exp}\, q$. Then

- $\underline{X}_{T_q}$ and $X_{T_q} - \underline{X}_{T_q}$ – independent;
- $\bar{X}_{T_q}$ and $X_{T_q} - \underline{X}_{T_q}$ – identically distributed.

Functions $\phi_q^{\pm}(\xi)$ – characteristic functions of distributions $P^{\pm}(dy)$, satisfying

$$\mathrm{supp} P^{+} \subset [0, +\infty), \mathrm{supp} P^{-} \subset (-\infty, 0].$$

Symbols of the EPV-operators

$$\frac{q}{q + \psi(\xi)} = e^{-i\xi x} \mathcal{E}_q e^{i\xi x},$$

$$\phi_q^{+}(\xi) = e^{-i\xi x} \mathcal{E}_q^{+} e^{i\xi x}, \quad \phi_q^{-}(\xi) = e^{-i\xi x} \mathcal{E}_q^{-} e^{i\xi x}.$$

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Wiener-Hopf factorization

- In Kuznetsov (2010), new classes of Lévy processes with explicit WH-factorization were suggested;
- In Eberlein et al. (2011) derive expressions for the analytically extended characteristic function of the supremum and the infimum of a Lévy process;

Bibliography

KUZNETSOV, A. Wiener–Hopf factorization and distribution of extrema for a family of Lévy processes. Ann. Appl. Probab. 20 (2010), no. 5, 1801–1830

EBERLEIN E., GLAU K., PAPAPANTOLEON A. Analyticity of the Wiener–Hopf factors and valuation of exotic options in Lévy models. In G. Di Nunno, B. Oksendal (Eds.), Advanced Mathematical Methods for Finance, pp. 223—245, Springer, 2011

"Wiener-Hopf Monte Carlo"

Kuznetsov et al. (2011) suggest a technique for simulating the joint law of the position and running maximum (minimum) at a fixed time of a general Lévy process. The WHMC-method

- is based on WH-factorization, stationarity and independence of Lévy process increments;
- uses time randomization;
- can be applied only if explicit formulas for WH-factors are known.

Bibliography

KUZNETSOV, A. AND KYPRIANOU, A. E. AND PARDO, J. C. AND VAN SCHAIK, K. A Wiener-Hopf Monte Carlo simulation technique for Lévy processes. *Ann. Appl. Probab.*, 2011, 21, 2171–2190.

The WHMC-method

Theorem 1, Kuznetsov et al (2011)

Set $q = n/T$. Let $\{S_q^j : j \geqslant 1\}$ and $\{I_q^j : j \geqslant 1\}$ be i.i.d. sequences of random variables with common distribution equal to that of $\overline{X}_{T_{(q)}}$ and $\underline{X}_{T_{(q)}}$, resp. Then, for all $n \in \mathbb{N}$,

$$(X_{\Gamma(n,q)}, \underline{X}_{\Gamma(n,q)}) \overset{d}{=} (V_q^n, J_q^n),$$

where, for any $k \in \mathbb{N}$, and setting $V_q^0 := 0$ and $J_q^0 := 0$ we define

$$V_q^k = V_q^{k-1} + \left(S_q^k + I_q^k\right), \quad J_q^k = \min\left\{J_q^{k-1}, V_q^{k-1} + I_q^k\right\}.$$

Application

$$\Gamma(n, n/T) \to T, \quad n \to +\infty \quad \text{a.s.}$$

$$(X_{\Gamma(n,n/T)}, \underline{X}_{\Gamma(n,n/T)}) \to (X_T, \underline{X}_T), \quad n \to +\infty \quad \text{a.s}$$

The problem for exotic options

We consider options, whose payoff at maturity date T depends on $(X_T, \underline{X}_T)$.

Consider

$$V(T,x) = E^x\left[g(X_T, \underline{X}_T)\right],$$

where
time 0 is the beginning of a period under consideration,
T – the final date,
$g(X_T, \underline{X}_T)$ – the payoff at time T.

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Time randomization and Laplace transform

Laplace transform

$$\begin{aligned}\hat{V}(q,x) &= \int_0^{+\infty} e^{-qt} E^x\left[g(X_t,\underline{X}_t)\right]dt \\ &= E\left[\int_0^{\infty} e^{-qt} g(x+X_t, x+\underline{X}_t)dt\right] \\ &= q^{-1}E[g(x+X_{T_q}, x+\underline{X}_{T_q})] \\ &= q^{-1}E[g(x+\bar{X}_{T_q}+\underline{X}_{T_q}, x+\underline{X}_{T_q})].\end{aligned}$$

$$\begin{aligned}\frac{(-1)^{n-1}q^n}{(n-1)!}\partial_q^{n-1}\hat{V}(q,x) &= \frac{q^n}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-qt}E^x\left[g(X_t,\underline{X}_t)\right]dt \\ &= E[g(x+X_{\Gamma(n,q)}, x+\underline{X}_{\Gamma(n,q)})].\end{aligned}$$

Numerical Laplace transform inversion

Post-Widder formula

If $f(\tau)$ is a function of a nonnegative real variable τ and the Laplace transform $\tilde{f}(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau)\, d\tau$ is known, the approximate Post-Widder formula for $f(\tau)$ can be written as

$$f(\tau) = \lim_{N\to\infty} f_N(\tau); \quad f_N(\tau) := \frac{(-1)^{N-1}}{(N-1)!} \left(\frac{N}{\tau}\right)^N \tilde{f}^{(N-1)}\left(\frac{N}{\tau}\right),$$

where $\tilde{f}^{(N)}(\lambda)$ – Nth derivative of the Laplace transform $\tilde{f}$ at λ. The convergence $f_N(\tau)$ to $f(\tau)$ as $N \to \infty$ is slow (of order N^{-1})

Barrier options in Lévy models, Post-Widder formula

KUDRYAVTSEV, O., "An efficient numerical method to solve a special class of integro-differential equations relating to the Levy models" // *Mathematical Models and Computer Simulations*, 2011, V.3., N.6., pp. 706-711.

Numerical Laplace transform inversion: the Gaver-Stehfest algorithm

An approximate formula for $f(\tau)$ can be written as follows

$$f(\tau) \approx \frac{1}{\tau} \sum_{k=1}^{N} \omega_k \cdot \tilde{f}\left(\frac{\alpha_k}{\tau}\right), \qquad 0 < \tau < \infty,$$

$$\begin{aligned} N &= 2n; \\ \alpha_k &= k \ln(2) \\ \omega_k &:= \frac{(-1)^{n+k} \ln(2)}{n!} \sum_{j=[(k+1)/2)]}^{\min\{k,n\}} j^{n+1} C_n^j C_{2j}^j C_j^{k-j}, \end{aligned}$$

where $[x]$ – integer part x и $C_L^K = \frac{L!}{(L-K)!K!}$ – binomial coefficients.

Approximate Wiener-Hopf factorization

The Fast Wiener-Hopf factorization method (FWHF-method)

- In Kudryavtsev and Levendorskiǐ (2009) the fast, accurate and universal numerical method for pricing barrier option under Lévy models was developed.
- In Kudryavtsev (2016) the approximate factorization was generalized; convergence of the method was accelerated.

Reference

Kudryavtsev, O.E., and S.Z. Levendorskiǐ, "Fast and accurate pricing of barrier options under Levy processes", *J. Finance and Stochastics*, 2009, V. 13, N. 4, 531-562

Kudryavtsev O. Advantages of the Laplace transform approach in pricing first touch digital options in Lévy-driven models. Boletín de la Sociedad Matemática Mexicana, 2016, vol. 22(2), pp. 711–731.

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Laplace transform of cdf functions

WH method

- Denote by $F_{\pm}(x, T)$ the cdf for $\bar{X}_T$ and $\underline{X}_T$
- Apply Laplace transform to $F_+(-x, T)$, $x < 0$:

$$\hat{F}_+(x, q) = \int_0^{+\infty} e^{-qt} E^x \left[\mathbf{1}_{(-\infty,0)}(\bar{X}_t) \right] dt = q^{-1} E \left[\mathbf{1}_{(-\infty,0)}(x + \bar{X}_{T_q}) \right]$$

- Apply Laplace transform to $F_-(-x, T)$, $x > 0$:

$$\hat{F}_-(x, q) = \int_0^{+\infty} e^{-qt} E^x \left[\mathbf{1}_{(-\infty,0)}(\underline{X}_t) \right] dt = q^{-1} E \left[\mathbf{1}_{(-\infty,0)}(x + \underline{X}_{T_q}) \right]$$

A generalization of the WHMC-method

Key ideas

- Approximate Wiener-Hopf factors $\phi_q^{\pm}(\xi)$ by using the FFT for real-valued functions.
- Apply the Laplace transform to $F_{\pm}(-x, T)$
- Find at q specified by the Gaver-Stehfest algorithm:

$$\hat{F}_+(x,q) = q^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,0)}(x) \quad \hat{F}_-(x,q) = q^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,0)}(x).$$

- Cdf $F_{\pm}(x)$ can be recovered from $\hat{F}_{\pm}(-x,q)$ by the Gaver-Stehfest algorithm.
- If the cdf F_X is known then one may simulate X by using samples from $F_X^{-1}(U)$, where U is a uniform distribution on $(0,1)$.

Approximate Wiener-Hopf factorization

A new formula for $\phi_q^+(\xi)$

Let a Lévy process X_t belongs to the class RLPE. Then there exists a constant $\omega_- < 0$ such that $\phi_q^+(\xi)$ admits analytical continuation into half-plane $\mathrm{Im}\,\xi > \omega_-$ and can be represented as follows:

$$\begin{aligned}
\phi_q^+(\xi) &= \exp\left[i\xi\Phi^+(0) - \xi^2\hat{\Phi}^+(\xi)\right], \\
\Phi^+(x) &= \mathbf{1}_{(-\infty,0]}(x)(2\pi)^{-1}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-} e^{ix\eta}\frac{\ln(q+\psi(\eta))}{\eta^2}d\eta; \\
\hat{\Phi}^+(\xi) &= \int_{-\infty}^{+\infty} e^{-ix\xi}\Phi^+(x)dx.
\end{aligned}$$

The FFT-based computation of $\phi_q^+(\xi)$ requires $O(M \ln M)$ operations, where M is a number of points for numerical integration

Approximate Wiener-Hopf factorization

A new formula for $\phi_q^-(\xi)$

Let a Lévy process X_t belongs to the class RLPE. Then there exists a constant $\omega_+ > 0$, such that $\phi_q^-(\xi)$ admits analytical continuation into half-plane $\mathrm{Im}\,\xi < \omega_+$ and can be represented as follows:

$$\begin{aligned}
\phi_q^-(\xi) &= \exp\left[-i\xi\Phi^-(0) - \xi^2\hat{\Phi}^-(\xi)\right], \\
\Phi^-(x) &= \mathbf{1}_{[0,+\infty)}(x)(2\pi)^{-1}\int_{-\infty+i\omega_+}^{+\infty+i\omega_+} e^{ix\eta}\frac{\ln(q+\psi(\eta))}{\eta^2}d\eta; \\
\hat{\Phi}^-(\xi) &= \int_{-\infty}^{+\infty} e^{-ix\xi}\Phi^-(x)dx.
\end{aligned}$$

The FFT-based computation of $\phi_q^-(\xi)$ requires $O(M\ln M)$ operations, where M is a number of points for numerical integration

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Risk measures

Risk measures as option prices

The key quantity of interest is the joint law of the current position and the running extrema of a Lévy process at a fixed time. The problem is closely related to pricing exotic options.

Illiquidity as an option

According to Longstaff (1995), an expected difference between the maximal stock price over the period and the price in the end of the period gives an upper bound for the value of the stock illiquidity.

Illiquidity as a lookback option

Consider an investor who cannot buy (sell) a stock during a certain time period. A floating strike European lookback call (put) gives the option holder the right to buy (sell) an asset at its lowest (highest) price during the life of the option.

Illiquidity as a lookback option

REPO (repurchase agreement)

In a repo, one party sells an asset (usually fixed-income securities) to another party at one price at the start of the transaction and commits to repurchase the fungible assets from the second party at a different price at a future date or (in the case of an open repo) on demand.

If the seller defaults during the life of the repo, the buyer (as the new owner) can sell the asset to a third party to offset his loss. The asset therefore acts as collateral and mitigates the credit risk that the buyer has on the seller.

During the life of the repo, the asset becomes illiquid for the seller. REPO operations are typically short-term. As an illiquidity risk was estimated before a repo, an investor should dynamically monitor the risk observing the asset prices.

Lookback options: floating strike

European floating strike lookback put

$$V(T,x) = E^x\left[e^{-rT}(Se^{\bar{X}_T} - Se^{X_T})\right],$$

Seasoned European floating strike lookback put

$$V(T_1, T_2; x, h) = E_{T_1}\left[e^{-r(T_2-T_1)}(Se^{\bar{X}_{T_2}} - Se^{X_{T_2}})|X_{T_1} = x, \bar{X}_{T_1} = h\right].$$

Set $T = T_2 - T_1$.

$$\begin{aligned} V(T,x) &= E^x\left[e^{-rT}S(e^{\max\{\bar{X}_T,h\}} - e^{X_T})\right] \\ &= E^x\left[e^{-rT}S(e^{\bar{X}_T} - e^{X_T})\right] + \\ &= E^x\left[e^{-rT}(H - Se^{\bar{X}_T})\mathbf{1}_{\{\bar{X}_T<h\}}\right]. \end{aligned}$$

$H(= Se^h)$ – predefined maximum, $E^x\left[e^{-rT}e^{X_T}\right] = e^x$.

Lookback options: fixed strike

European fixed strike lookback put

$$V(T,x) = E^x\left[e^{-rT}(K - Ke^{\underline{X}_T})_+\right],$$

Seasoned European fixed strike lookback put

$$V(T_1, T_2; x, h) = E_{T_1}\left[e^{-r(T_2-T_1)}(K - Ke^{\underline{X}_{T_2}})_+ | X_{T_1} = x, \underline{X}_{T_1} = h\right].$$

Set $T = T_2 - T_1$.

$$\begin{aligned} V(T,x) &= E^x\left[e^{-rT}(K - Ke^{\min\{\underline{X}_T, h\}})_+\right] \\ &= E^x\left[e^{-rT}(K - Ke^{\underline{X}_T})_+\right] + \\ &= E^x\left[e^{-rT}\left((H - Ke^{\underline{X}_T})_+ - (K - Ke^{\underline{X}_T})_+\right)\mathbf{1}_{\{\underline{X}_T > h\}}\right]. \end{aligned}$$

$H(= Ke^h)$ – predefined minimum

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gAWHF&MC-method

The algorithm of the generalized approximate Wiener-Hopf factorization Monte Carlo method was published in Kudryavtsev O.E.(2019). We will refer to it as gAWHF&MC-method.

Bibliography

KUDRYAVTSEV, O.E., “Approximate Wiener–Hopf factorization and Monte Carlo methods for Lévy processes”, *Theory Probab. Appl.*, 2019, Vol. 64, No. 2, to appear

Numerical examples

We check the performance of the gAWHF&MC-method against prices obtained by deterministic methods: the FWHF&GS-method from Kudryavtsev O., Levendorskii S. (2011) and the ParaiLT-method from Boyarchenko S. I., Levendorskii S. Z.(2013).

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KUDRYAVTSEV, O.E., AND S.Z. LEVENDORSKIĬ, "Efficient pricing options with barrier and lookback features under Levy processes", Working paper, 2011, 29 pp. Available at SSRN.

BOYARCHENKO S. I., LEVENDORSKII S. Z. "Efficient Laplace inversion, Wiener-Hopf factorization and pricing lookbacks", *International Journal of Theoretical and Applied Finance*, 2013, vol. 16(3), 1350011.

Experiment setup

We consider European fixed strike lookback put options under the TSL model, and use the same parameters of the KoBoL (CGMY) process as in Boyarchenko S. I., Levendorskii S. Z.(2013)://
$c = 0.2395$, $\lambda_+ = 3.0$, $\lambda_- = -10.0$, $\nu = 1.2$ ($C = 0.2395$, $G = 3.0$, $M = 10.0$, $Y = 1.2$ in CGMY parametrization).

The remaining parameters are strike price $K = 100$, the dividend rate $d = 0$ and interest rate $r = 0.04$. The drift parameter μ is fixed by EMM-requirement. We consider 2 maturities $T = 0.1$ (short) and $T = 2$ (long).

The computations performed in 10 points
$x_k = \ln(S/K)(= 0.02; 0.04; ...; 0.2)$, where S – initial spot price.

PC characteristics: Intel Core(TM)i5 CPU, 1.7GHz, 4 GB RAM, Windows 7 Professional with 64-bit

Numerical examples

European lookbacks

For verification of the accuracy of our method, we calculate prices for the fixed strike lookback put by using the gAWHF&MC-method, the FWHF&GS-method from the ParaiLT-method.

The prices of the FWHF&GS-method were obtained using the code implemented into the program platform Premia (www.premia.fr).

The prices of the ParaiLT-method were taken from the table 3 Boyarchenko S. I., Levendorskii S. Z.(2013), as well as the benchmarks.

gAWHF&MC-prices converge very fast and agree with the benchmarks. All the methods are in agreement. gAWHF&MC-method for pricing lookback options could be considered as a competitor to the deterministic methods.

Convergence of gAWHF&MC

Fixed strike lookback put prices and MC-errors. Short maturity

Parameters	$h=0.001,\ N=10^4$		$h=0.001,\ N=10^5$		$h=0.001,\ N=10^6$	
$x=\ln(S/K)$	price	error	price	error	price	error
0.02	5.34748	0.147	5.38840	0.046	5.39141	0.015
0.04	4.21819	0.140	4.25655	0.044	4.26282	0.014
0.06	3.34641	0.132	3.37984	0.042	3.38694	0.013
0.08	2.67288	0.123	2.69821	0.039	2.70609	0.012
...	...	...	...	...	...	...
0.2	0.81726	0.079	0.81431	0.025	0.81986	0.008
Parameters	$h=0.0005,\ N=10^4$		$h=0.0005,\ N=10^5$		$h=0.0005,\ N=10^6$	
$x=\ln(S/K)$	price	error	price	error	price	error
0.02	5.33876	0.147	5.37970	0.046	5.38274	0.015
0.04	4.21152	0.140	4.24982	0.044	4.25610	0.014
0.06	3.34122	0.132	3.37463	0.042	3.38172	0.013
0.08	2.66890	0.123	2.69414	0.039	2.70203	0.012
...	...	...	...	...	...	...
0.2	0.81631	0.079	0.81333	0.025	0.81888	0.008
Parameters	$h=0.0001,\ N=10^4$		$h=0.0001,\ N=10^5$		$h=0.0001,\ N=10^6$	
$x=\ln(S/K)$	price	error	price	error	price	error
0.02	5.33171	0.147	5.37264	0.046	5.37571	0.015
0.04	4.20611	0.140	4.24435	0.044	4.25064	0.014
0.06	3.33699	0.132	3.37037	0.042	3.37745	0.013
0.08	2.66565	0.123	2.69079	0.039	2.69869	0.012
...	...	...	...	...	...	...
0.2	0.81547	0.079	0.81246	0.025	0.81801	0.008

Convergence of gAWHF&MC

Fixed strike lookback put prices and MC-errors. Long maturity

Parameters	$h = 0.001,\ N = 10^5$		$h = 0.001,\ N = 10^6$		$h = 0.001,\ N = 10^7$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	28.29943	0.125	28.27669	0.040	28.26846	0.013
0.04	27.15904	0.126	27.13597	0.040	27.12789	0.013
0.06	26.05742	0.127	26.03453	0.040	26.02668	0.013
0.08	24.99020	0.128	24.96794	0.040	24.96014	0.013
...	...	...	...	...	...	...
0.2	19.22577	0.125	19.21361	0.040	19.20667	0.013
Parameters	$h = 0.0005,\ N = 10^5$		$h = 0.0005,\ N = 10^6$		$h = 0.0005,\ N = 10^7$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	28.29156	0.125	28.26881	0.040	28.26058	0.013
0.04	27.15145	0.126	27.12838	0.040	27.12031	0.013
0.06	26.05008	0.127	26.02719	0.040	26.01934	0.013
0.08	24.98308	0.128	24.96082	0.040	24.95302	0.013
...	...	...	...	...	...	...
0.2	19.21983	0.125	19.20769	0.040	19.20075	0.013
Parameters	$h = 0.0001,\ N = 10^5$		$h = 0.0001,\ N = 10^6$		$h = 0.0001,\ N = 10^7$	
$x = \ln(S/K)$	price	error	price	error	price	error
0.02	28.28453	0.125	28.26178	0.040	28.25354	0.013
0.04	27.14467	0.126	27.12160	0.040	27.11352	0.013
0.06	26.04351	0.127	26.02061	0.040	26.01277	0.013
0.08	24.97671	0.128	24.95444	0.040	24.94665	0.013
...	...	...	...	...	...	...
0.2	19.21448	0.125	19.20235	0.040	19.19541	0.013

Comparison of gAWHF&MC with deterministic methods

Errors. Short maturity

x	0.02	0.04	0.06	0.08	...	0.20	Time
V_{put}	5.37205	4.24803	3.37586	2.69765	...	0.81512	400-1900
Para iLT							
ϵ=E-01	-0.053	-0.038	-0.028	-0.014	...	0.0064	0.03-0.15
ϵ=E-02	-0.0054	-0.0041	-0.0032	-0.0025	...	-0.0007	0.55-1.49
FWHF&GS_7							
$h = 0.001$	0.00669	0.00633	0.00546	0.00447	...	0.00094	0.078
$h = 0.0005$	0.00352	0.00326	0.00276	0.00223	...	0.00043	0.188
$h = 0.0002$	0.00124	0.00111	0.00091	0.00072	...	0.00009	0.39
gAWHF&MC							
$h = 0.001$							
$N = 10^4$	-0.0246	-0.02984	-0.02945	-0.02478	...	0.00215	0.156
$N = 10^5$	0.0164	0.00852	0.00398	0.00055	...	-0.00081	0.1880
$N = 10^6$	0.0194	0.01479	0.01108	0.00843	...	0.00474	0.797
$h = 0.0005$							
$N = 10^4$	-0.0333	-0.03650	-0.03464	-0.02875	...	0.00120	0.266
$N = 10^5$	0.0077	0.00180	-0.00123	-0.00352	...	-0.00178	0.328
$N = 10^6$	0.0107	0.00808	0.00586	0.00438	...	0.00377	0.906
$h = 0.0001$							
$N = 10^4$	-0.0403	-0.04191	-0.03887	-0.03201	...	0.00036	1.125
$N = 10^5$	0.0006	-0.00367	-0.00548	-0.00686	...	-0.00266	1.172
$N = 10^6$	0.0037	0.00262	0.00160	0.00104	...	0.00289	1.813

Comparison of gAWHF&MC with deterministic methods

Errors. Long maturity

x	0.02	0.04	0.06	0.08	...	0.20	Time
V_{put}	28.25454	27.11439	26.01360	24.94750	...	19.19671	873-2762
Para iLT							
ϵ=E-03	-0.068	-0.062	-0.057	-0.052	...	-0.029	0.23-1.13
ϵ=E-04	-0.0043	-0.0058	-0.0060	-0.0057	...	-0.0033	1.36-5.1
FWHF&GS_7							
$h = 0.001$	0.02686	0.02731	0.02770	0.02790	...	0.0281	0.093
$h = 0.0005$	0.01266	0.01291	0.01300	0.01310	...	0.0132	0.187
$h = 0.0002$	0.00196	0.00221	0.00230	0.00250	...	0.0030	0.359
gAWHF&MC							
$h = 0.001$							
$N = 10^5$	0.0449	0.0446	0.0438	0.0427	...	0.0291	0.218
$N = 10^6$	0.0222	0.0216	0.0209	0.0204	...	0.0169	1.062
$N = 10^7$	0.0139	0.0135	0.0131	0.0126	...	0.0100	7.59
$h = 0.0005$							
$N = 10^5$	0.0370	0.0371	0.0365	0.0356	...	0.0231	0.359
$N = 10^6$	0.0143	0.0140	0.0136	0.0133	...	0.0110	1.062
$N = 10^7$	0.0060	0.0059	0.0057	0.0055	...	0.0040	7.98
$h = 0.0001$							
$N = 10^5$	0.0300	0.0303	0.0299	0.0292	...	0.0178	1.156
$N = 10^6$	0.0072	0.0072	0.0070	0.0069	...	0.0056	1.859
$N = 10^7$	-0.0010	-0.0009	-0.0008	-0.0009	...	-0.0013	9.53